Inverting Quantum Decoherence by Classical Feedback from the Environment

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We show that for qubits and qutrits it is always possible to perfectly recover quantum coherence by performing a measurement only on the environment, whereas for dimension $d > 3$ there are situations where recovery is impossible, even with complete access to the environment. For qubits, the minimal amount of classical information to be extracted from the environment equals the *entropy exchange*.

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Decoherence is universally considered, on one side, as the major practical limitation for communication and processing of quantum information. On the other side, decoherence yields the key concept to explain the transition from quantum to classical world [1] due to the uncontrolled and unavoidable interactions with the environment. Great effort in the literature has been devoted to combat the effect of decoherence by engineering robust encoding-decoding schemes: by storing quantum information on a larger Hilbert space—as in quantum error correction [2]—or by protecting the quantum information in a decoherence-free subspace [3], or else via topological constraints [4]. Moreover, some authors have recently addressed a different approach to undo quantum noises by extracting classical information from the environment [5] and exploiting it as an additional amount of side information useful to improve quantum communication performances [6].

The recovery of quantum coherence from the environment is often a difficult task, e.g., when the environment is "too big" to be controlled, as for spontaneous emission of radiation [7]. By regaining control on the environment the recovery can sometimes be actually accomplished, for example, by keeping the emitted radiation inside a cavity. However, in some cases, the full recovery of quantum coherence becomes impossible even in principle, namely, even when one has complete access to the environment. This naturally leads us to pose the following question: in which physical situations is it possible to perfectly recover quantum coherence by monitoring the environment?

In this Letter we will show that for qubits and qutrits it is always possible to perfectly cancel the effect of decoherence by monitoring—i.e., measuring—the environment. On the contrary, for quantum systems with larger dimension d, namely, for qudits with $d > 3$, there are situations where the recovery is impossible even in principle. In order to prove the above assertion we will give a complete classification of decoherence maps for any finite dimension d, showing that they are always of the form of a Schur map. For qubits we will also evaluate the minimal amount of classical information that must be extracted from the environment in order to invert the decoherence process.

A completely decohering evolution asymptotically cancels any quantum superposition when reaching the stationary state, making any state diagonal in some fixed orthonormal basis—the basis depending on the particular system-environment interaction. In the Heisenberg picture we say that such a completely decohering evolution asymptotically maps the whole algebra of quantum observable into a "maximal classical algebra," that is a maximal set of commuting—namely jointly measureable—observables. It is also possible to consider partial decoherence, i.e., preserving superpositions within subspaces of the total system, reducing the initial state into a block-diagonal form. Here, however, we focus our attention on the worst case scenario of complete decoherence, and also show how the present results can be easily generalized to partial decoherence.

Let us denote by $\mathcal{A}_q$ the "quantum algebra" of all bounded operators on the finite dimensional Hilbert space $\mathcal{H}$, and by $\mathcal{A}_c$ the "classical algebra", namely, any maximal Abelian subalgebra $\mathcal{A}_c \subset \mathcal{A}_q$. Clearly, all operators in $\mathcal{A}_c$ can be jointly diagonalized on a common orthonormal basis, which in the following will be denoted as $\mathsf{B} = \{|k\rangle | k = 1, \ldots, d\}$. Then, the classical algebra $\mathcal{A}_c$ is also the linear span of the one-dimensional projectors $|k\rangle\langle k|$, whence $\mathcal{A}_c$ is a d-dimensional vector space. According to the above general framework, we call *(complete) decoherence map* a completely positive identity-preserving (i.e., trace-preserving in the Schrödinger picture) map $\mathcal{E}$ which asymptotically maps any observable $O \in \mathcal{A}_q$ to a corresponding "classical observable" in $\mathcal{A}_c$, namely, such that the limit $\lim_{n\to\infty} \mathcal{E}^n(O)$ exists and belongs to the classical algebra $\mathcal{A}_c$ for any $O \in \mathcal{A}_q$. Here we denote with $\mathcal{E}^n$ the nth iteration of the map $\mathcal{E}$, implicitly assuming Markovian evolution.

It is easy to see that the set of decoherence maps is convex (i.e., if we mix two decoherence maps we obtain again a decoherence map). Such a convex set will be denoted by D. Moreover, D is a subset of the convex set of maps that preserve all elements of the classical algebra $\mathcal{A}_c$. Such maps have a remarkably simple form:

Theorem 1 (Schur form).—A map $\mathcal{E}$ preserves all elements of the maximal classical algebra if and only if it has the form

$$\mathcal{E}(O) = \xi \circ O. \tag{1}$$

$A \circ B$ denoting the Schur product of operators A and B, i.e., $A \circ B \equiv \sum_{k,l=1}^{d} A_{kl} B_{kl} |k\rangle\langle l|$, $\{A_{kl}\}$ and $\{B_{kl}\}$ being the matrix elements of A and B in the basis B, and ξ_{kl} being a correlation matrix, i.e., a positive semidefinite matrix with $\xi_{kk} = 1$ *for all* $k = 1, \ldots, d$.

Proof.—Consider a Kraus representation of the map $\mathcal{E}$:

$$\mathcal{E}(O) = \sum_{i=1}^{r} E_i^\dagger O E_i. \tag{2}$$

Exploiting a result by Lindblad [8], we know that a map $\mathcal{E}$ preserves all elements of an algebra $\mathcal{A}$ if and only if its Kraus operators commute with the algebra itself, i.e., $[E_i, O] = 0$ for any $O \in \mathcal{A}$. Since in our case the algebra is the maximal Abelian algebra $\mathcal{A}_c$, such a commutation relation implies $E_i \in \mathcal{A}_c$, therefore

$$E_i = \sum_{k=1}^{d} e_k^{(i)} |k\rangle\langle k|. \tag{3}$$

Substituting Eq. (3) into Eq. (2), we obtain

$$\mathcal{E}(O) = \sum_{k,l=1}^{d} \xi_{kl} O_{kl} |k\rangle\langle l|, \tag{4}$$

where

$$\xi_{kl} \equiv \sum_{i=1}^{r} e_k^{(i)*} e_l^{(i)}. \tag{5}$$

By definition, the matrix ξ_{kl} is positive semidefinite, and the identity-preserving condition $\mathcal{E}(1) = 1$ in Eq. (4) gives $\xi_{kk} = 1$ for all k. Vice versa, it is obvious to see that any map of the form (1) preserves all elements of the classical algebra. □

In the case of partial decoherence, the Shur form (1) generalizes to $\mathcal{E}(O) = \sum_{k,l} \xi_{kl} P_k O P_l$, where P_k's are the orthogonal projections over the invariant subspaces.

Since there is a linear correspondence between maps preserving $\mathcal{A}_c$ and correlation matrices, the two sets share the same convex structure, whence the map is extremal if and only if its correlation matrix is extremal.

Thanks to Theorem 1 the general form of a decoherence map is immediately recognizable:

Corollary 1.—A map $\mathcal{E}$ is a decoherence map if and only if it has the form (1) where ξ_{kl} is a correlation matrix with $|\xi_{kl}| < 1$ for all $k \neq l$.

Notice that positivity of ξ implies $|\xi_{kl}| \leq 1$, while the requirement that $\lim_{n\to\infty} \mathcal{E}^n(\mathcal{A}_q) = \mathcal{A}_c$ needs $|\xi_{kl}| < 1$ strictly. This also implies the following:

Corollary 2.—The closure $\bar{\mathsf{D}}$ of the set D of decoherence maps coincides with the set of maps that preserve the classical algebra.

As examples of maps on the border of $\bar{\mathsf{D}}$, simply consider the identity map, or the map $\mathcal{U}(\cdot) = U^\dagger \cdot U$, with the unitary U diagonal on the basis B.

Another relevant property of the decoherence maps is the following:

Corollary 3.—All decoherence maps commute among themselves; i.e., their order is irrelevant.

The action of a decoherence map on quantum states is given in Schrödinger picture by

$$\mathcal{E}_S(\rho) = \xi^T \circ \rho, \tag{6}$$

where T denotes transposition with respect to the basis B (also ξ^T is a correlation matrix). As a consequence, one has exponential decay of the off-diagonal elements of ρ, since $|[\mathcal{E}_S^n(\rho)]_{kl}| = |\xi_{lk}|^n \cdot |\rho_{kl}|$. In other words, any initial state ρ decays exponentially towards the completely decohered state $\sum_k \rho_{kk} |k\rangle\langle k| \equiv \rho_\infty$.

Lemma 1.—A map $\mathcal{E}$ is extremal in $\bar{\mathsf{D}}$ if and only if it is extremal in the set of all maps.

Proof.—Take $\mathcal{E}$ extremal in $\bar{\mathsf{D}}$, and suppose by contradiction that in the set of all maps there are two maps $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $\mathcal{E} = p\mathcal{E}_1 + (1-p)\mathcal{E}_2$. Since $\mathcal{E}$ leaves all elements of $\mathcal{A}_c$ invariant, for all k one has

$$|k\rangle\langle k| = \mathcal{E}(|k\rangle\langle k|) = p\mathcal{E}_1(|k\rangle\langle k|) + (1-p)\mathcal{E}_2(|k\rangle\langle k|). \tag{7}$$

But $\mathcal{E}_1$ and $\mathcal{E}_2$ are positive and identity-preserving maps, whence necessarily $\mathcal{E}_i(|k\rangle\langle k|) = |k\rangle\langle k|$ for $i = 1, 2$ and for all k, namely $\mathcal{E}_1$ and $\mathcal{E}_2$ are both in $\bar{\mathsf{D}}$. But $\mathcal{E}$ is extremal in $\bar{\mathsf{D}}$, whence $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$, and $\mathcal{E}$ is extremal in the set of all maps. The converse direction is trivial. □

As a consequence of Lemma 1, the convex structure of decoherence maps can be obtained by application of the well known Choi Theorem [9], which states that the canonical Kraus operators $\{E_i\}$, $1 \leq i \leq r$, of every extremal map are such that their products $\{E_i^\dagger E_j\}$, $1 \leq i, j \leq r$, are linearly independent. A relevant consequence of this characterization is the following

Theorem 2.—If $\mathcal{E} \in \bar{\mathsf{D}}$ is extremal, then $r \leq \sqrt{d}$. For qubits and qutrits any map in D is random-unitary.

Proof.—Because of linear independence, the dimension of the linear span of $\{E_i^\dagger E_j\}$ must be r^2. But this set of operators is a subset of $\mathcal{A}_c$, whose dimension is d. Therefore $r^2 \leq d$. In particular, for $d \leq 3$ one necessarily has $r = 1$; i.e., the extremal points of $\bar{\mathsf{D}}$ are unitary maps. □

This means that for qubits and qutrits every decoherence map can be written as

$$\mathcal{E}(O) = \sum_i p_i U_i^\dagger O U_i, \tag{8}$$

for some commuting unitary operators $U_i \in \mathcal{A}_c$ and probability distribution p_i. Now, in Ref. [5] it is shown that the only channels that can be perfectly inverted by monitoring the environment are the random-unitary ones. Therefore, it follows that one can perfectly correct any decoherence map for qubits and qutrits by monitoring the environment. The correction is achieved by retrieving the index i in Eq. (8) via a measurement on the environment, and then by applying the inverse of the unitary transformation U_i on the system (for pure joint system environment states the unitary form of the conditioned system transformations also follows from the Lo-Popescu Theorem [10] for local operations with classical communication). Therefore, the random-unitary map simply leaks $H(p_i)$ bits of classical information into the environment (H denoting the Shannon entropy), and the effects of decoherence can be completely eliminated by recovering such classical information, without any prior knowledge about the input state.

The fact that decoherence maps are necessarily random unitary is true only for qubits and qutrits. Indeed, for dimension $d \geq 4$ there are decoherence maps which are not random unitary, since there exist extremal correlation matrices whose rank is greater than one [11], e.g., the rank-two matrix

$$\xi = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & \frac{1+i}{2} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & \frac{1-i}{2} & 1 \end{pmatrix}. \tag{9}$$

The canonical Kraus decomposition of the map $\mathcal{E}(O) = \xi \circ O$, can be obtained by diagonalizing the operator ξ as $\xi = |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|$, with $\langle v_1|v_2\rangle = 0$. Then, the canonical Kraus operators are $E_i = \sum_k \langle k|v_i\rangle |k\rangle\langle k|$, $i = 1, 2$, and the corresponding map $\mathcal{E} = \sum_i E_i^\dagger \cdot E_i$ is not unitary (its canonical Kraus decomposition contains two terms), nor is random unitary, since it is extremal. Such decoherence maps with $r \geq 2$ represent a process which is fundamentally different from the random-unitary one, corresponding to a *leak of quantum information* from the system to the environment, information that cannot be perfectly recovered from the environment [5].

Now we address the problem of estimating the amount of classical information needed in order to invert a random-unitary decoherence map. If the environment is initially in a pure state, say $|0\rangle_e$, a useful quantity to deal with is the so-called entropy exchange [12] S_{ex} defined as

$$S_{\mathrm{ex}}(\rho) = S(\sigma_e^\rho), \tag{10}$$

where σ_e^ρ is the reduced environment state after the interaction with the system in the state ρ, and $S(\rho) = -\operatorname{Tr}[\rho \log \rho]$ is the von Neumann entropy. In the case of initially pure environment, the entropy exchange depends only on the map $\mathcal{E}$ and on the input state of the system ρ, regardless of the particular system-environment interaction chosen to model $\mathcal{E}$. It quantifies the information flow from the system to the environment and, for all input states ρ, one has the bound [12] $|S(\mathcal{E}_S(\rho)) - S(\rho)| \leq S_{\mathrm{ex}}(\rho)$, namely, the entropy exchange S_{ex} bounds the entropy production at each step of the decoherence process.

In order to explicitly evaluate the entropy exchange for a decoherence process, we can then exploit a particular model interaction between system and environment. This can be done starting from Eq. (5) noticing that it is always possible to write $\xi_{kl} = \langle e_k|e_l\rangle$ for a suitable set of normalized vectors $\{|e_k\rangle\}$. Then, the map $\mathcal{E}_S(\rho) = \xi^T \circ \rho$ can be realized as $\mathcal{E}_S(\rho) = \operatorname{Tr}_e[U(\rho \otimes |0\rangle\langle 0|_e)U^\dagger]$, where the unitary interaction U gives the transformation

$$U|k\rangle \otimes |0\rangle_e = |k\rangle \otimes |e_k\rangle. \tag{11}$$

The final reduced state of the environment is then $\sigma_e^\rho = \sum_k \rho_{kk} |e_k\rangle\langle e_k|$. Then, in order to evaluate S_{ex} for a decoherence map $\mathcal{E}_S(\rho) = \xi^T \circ \rho$, it is possible to bypass the evaluation of the states $|e_i\rangle$ of the environment, using the formula

$$S_{\mathrm{ex}}(\rho) = S(\sqrt{\rho_\infty}\,\xi\,\sqrt{\rho_\infty}), \tag{12}$$

which follows immediately from the fact that $\sqrt{\rho_\infty}\,\xi\,\sqrt{\rho_\infty}$, and σ_e^ρ are both reduced states of the same bipartite pure state $\sum_i \sqrt{\rho_{ii}} |i\rangle|e_i\rangle$.

The unitary interaction U in Eq. (11) generalizes the usual form considered for quantum measurements [13], with the quantum system interacting with a pointer, which is left in one of the (nonorthogonal) states $\{|e_k\rangle\}$. The more the pointer states are classical—i.e., distinguishable—the larger is the entropy exchange, and, therefore, the faster the decoherence process. In the limit of orthogonal states, decoherence is instantaneous; i.e., $\mathcal{E}_S(\rho) = \rho_\infty$. If the state of the system is a pure classical one $\rho = |j\rangle\langle j|$, the entropy exchange is zero, since $\sigma_e^\rho = |e_j\rangle\langle e_j|$ is pure. In this case the environment evolves freely, with the system untouched. For mixed classical state ρ there is a nonvanishing entropy flow, even if the state of the system does not change. This is because the entropy flow is well defined only for a closed system—i.e., described by a unitarily evolving global pure state, with $\Delta S_{\mathrm{tot}} = 0$—whence in the entropic balance one must consider also a reference system r purifying ρ. As an example, let $\mathcal{A}_c \ni \rho = \sum_i p_i |i\rangle\langle i|$ be purified as $|\Psi\rangle = \sum_i \sqrt{p_i} |i\rangle_r |i\rangle$. Then, the action of $1_r \otimes U$ on $|\Psi\rangle|0\rangle_e$ is $(1_r \otimes U)|\Psi\rangle|0\rangle_e = \sum_i \sqrt{p_i} |i\rangle_r |i\rangle|e_j\rangle$ and the reduced reference + system state changes according to

$$|\Psi\rangle\langle\Psi| \mapsto R = \sum_{ij} \xi_{ji} \sqrt{p_i p_j} |i\rangle\langle j|_r \otimes |i\rangle\langle j|, \tag{13}$$

corresponding to $\mathcal{E}_S(\rho) = \operatorname{Tr}_r[R] \equiv \rho$. In other words, the non-null entropy exchange results in a decrease of the correlations between the reference and the system.

When a map can be inverted by monitoring the environment—i.e., in the random-unitary case—the entropy exchange $S_{\mathrm{ex}}(1/d)$ provides a lower bound to the amount of classical information that must be collected from the environment in order to perform the correction scheme of Ref. [5]. In fact, assuming a random-unitary decomposition (8) and using the formula [12] $S_{\mathrm{ex}}(\rho) = S(\sum_{i,j} \sqrt{p_i p_j} \operatorname{Tr}[U_i \rho U_j^\dagger] |i\rangle\langle j|)$, we obtain

$$S_{\rm ex}(1/d) \leq H(p_i). \tag{14}$$

The inequality comes from the fact that the diagonal entries of a density matrix are always majorized by its eigenvalues, and it becomes equality if and only if $\mathrm{Tr}[U_i U_j^\dagger]/d = \delta_{ij}$; i.e., the map admits a random-unitary decomposition with *orthogonal* unitary operators. Moreover, from Eq. (12) we have $S_{\rm ex}(1/d) = S(\xi/d)$.

For qubits, $S(\xi/d)$ quantifies exactly the minimum amount of classical information which must be extracted from the environment. Indeed, a unitary map in $\bar{\mathsf{D}}$ corresponds to the rank-one correlation matrix $\xi = |\phi\rangle\langle\phi|$, where

$$|\phi\rangle = \sum_{k=1}^{d} e^{i\phi_k}|k\rangle. \tag{15}$$

For $d = 2$ it is simple to see that any correlation matrix ξ can be diagonalized using two such vectors, i.e., $\xi = p_1|\phi_1\rangle\langle\phi_1| + p_2|\phi_2\rangle\langle\phi_2|$, whence the corresponding map is random-unitary with $U_i = \Sigma_k \langle k|\phi_i\rangle|k\rangle\langle k|$. Clearly, $\langle\phi_1|\phi_2\rangle = 0$ implies $\mathrm{Tr}[U_1^\dagger U_2] = 0$. Therefore for qubits $H(p_i) = S(\xi/d)$.

Notice that the same decoherence map may be obtainable by a random-unitary transformation with more than two outcomes, and a flattened probability distribution $\{p_i\}$, corresponding to a larger information $H(p_i)$. However, for qubits it is always possible to perform a suitable measurement on the environment and to invert the decoherence map retrieving the minimal amount of information from the environment.

For dimension $d > 2$, the bound in Eq. (14) is generally strict. Already for dimension $d = 3$, even if all decoherence maps are random unitary, the amount of information required for perfect correction may exceed $S(\xi/d) = S_{\rm ex}(1/d)$. As an example, the correlation matrix with nondegenerate spectrum

$$\xi = \begin{pmatrix} 1 & 0 & 1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{pmatrix}. \tag{16}$$

has the eigenvector $|v\rangle = |1\rangle - |2\rangle$, which is not of the form (15). This means that it is not possible to write $\mathcal{E}_S(\rho) = \xi^T \circ \rho$ as a convex combination of orthogonal unitaries.

Finally, it is worth noticing that, when a decoherence process can be inverted, this can be done regardless of the number of iterations of the map, since the iterated map is also random unitary. Clearly one needs to perform the measurement on a larger Hilbert space for the environment; however, the complexity of the measurement does not necessarily increase. In fact, in order to restore the initial state we only need to know how many times the unitary U_i for each i has been applied to the system, since the unitary operators for different i commute and their order is irrelevant.

In summary, in this Letter we have shown that for qubits and qutrits it is always possible to perfectly invert decoherence by extracting classical information from the environment. For dimension $d = 4$, instead, we gave a counterexample proving that for $d > 3$ generally the recovery is impossible even with complete access to the environment. A complete classification of decoherence maps for any finite dimension d has been given in form of a Schur product with a correlation matrix ξ. The minimal amount of classical information needed to invert decoherence has been evaluated for qubits as the von Neumann entropy of ξ/d.

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