

Superbroadcasting of continuous variable mixed states

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Abstract. We consider the problem of broadcasting quantum information encoded in the average value of the field from N to $M > N$ copies of mixed states of radiation modes. We derive the broadcasting map that preserves the complex amplitude, while optimally reducing the noise in conjugate quadratures. We find that from two input copies broadcasting is feasible, with the possibility of simultaneous purification (*superbroadcasting*). We prove similar results for purification ($M \leq N$) and for phase-conjugate broadcasting.

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1. Introduction

Quantum cloning is impossible [1]. This means that one cannot produce a number of independent physical systems prepared in identical states out of a smaller amount of systems prepared in the same state. Since the formulation of the no-cloning theorem, the search for quantum devices that can perform cloning with the highest possible fidelity gave rise to a whole branch in the literature. Optimal cloners have been found, for qubits [2–4], for general finite-dimensional systems [5], for restricted sets of input states [6, 7], and for infinite-dimensional systems such as harmonic oscillators—the so-called continuous variable cloners [8]. However, for the case of mixed states, a different type of cloning transformation can be considered—the so-called *broadcasting*—in which the output copies are in a globally correlated state whose local ‘reduced’ states are identical to the input states. This possibility has been considered in [9], where it has been shown that broadcasting a single copy from a non-commuting set of density matrices is always impossible. Later, such a result has been considered in the literature as the generalization of the no-cloning theorem to mixed states. However, more recently, for qubits an effect called *superbroadcasting* [10] has been discovered, which consists of the possibility of broadcasting the state while even increasing the purity of the local state, for at least $N \geq 4$ input copies, and for sufficiently short input Bloch vector (and even for $N = 3$ input copies for phase-covariant broadcasting instead of universal covariance [11]).

In the present paper, we analyse the broadcasting of continuous variable mixed states by a signal-preserving map. More precisely, this means that we consider a set of states obtained by displacing a fixed mixed state by a complex amplitude in the harmonic oscillator phase space, while the broadcasting map is covariant with respect to the (Weyl–Heisenberg) group of complex displacements. We will focus mainly on displaced thermal states (which are equivalent to coherent states that have suffered Gaussian noise); however, all results of the present paper hold in terms of noise of conjugated quadratures for the set of states obtained by displacing any fixed state.

As we will see, superbroadcasting is possible for continuous variable mixed states, namely one can produce a larger number of copies, which are purified locally on each use, and with the same signal of the input. For displaced thermal states, for example, superbroadcasting can be achieved for at least $N = 2$ input copies, with thermal photon number $\bar{n}_{\text{in}} \geq 1/3$, whereas, for sufficiently large \bar{n}_{in} at the input, one can broadcast to an unbounded number M of output copies. For purification (i.e. $M \leq N$), quite surprisingly the purification rate is $\bar{n}_{\text{out}}/\bar{n}_{\text{in}} = N^{-1}$, independent of M . The particular case of 2 to 1 for noisy coherent states has been reported in [12]. We will also prove similar results for broadcasting of phase-conjugated copies of the input.

The paper is organized as follows. In section 2 we introduce the problem of covariant broadcasting, deriving the general form of a covariant channel (trace-preserving completely positive (CP) map), and introduce a special channel that broadcasts from N to $M > N$ copies. In section 3, we prove that such a channel is optimal for broadcasting any noisy displaced state. In section 4, we consider the same problem for purification (i.e. $M < N$). In section 5, we derive superbroadcasting for the output copies with a conjugate phase with respect to the originals. In section 6, we show the optimality by a simpler derivation, namely by exploiting the bounds from the theory of linear amplification (which is then based on supplementary assumptions). In section 7, we show a simple experimental scheme to achieve optimal broadcasting/purification. Section 8 closes the paper with a summary of results and some concluding remarks.

2. Covariant broadcasting for the Weyl–Heisenberg group

We consider the problem of broadcasting N input copies of displaced (generally) mixed states of harmonic oscillators (with boson annihilation operators denoted by a_0, a_1, \dots, a_{N-1}) to M output copies (with boson annihilation operators b_0, b_1, \dots, b_{M-1}). In order to preserve the signal, the broadcasting map B must be covariant, i.e. in formula

$$B(D(\alpha)^{\otimes N} \Xi D(\alpha)^{\dagger \otimes N}) = D(\alpha)^{\otimes M} B(\Xi) D(\alpha)^{\dagger \otimes M}, \quad (1)$$

where $D_c(\alpha) = \exp(\alpha c^\dagger - \alpha^* c)$ denotes the displacement operator, and Ξ represents an arbitrary N -partite state. It is useful to consider the Choi–Jamiołkowski bijective correspondence of CP maps B from \mathcal{H}_{in} to \mathcal{H}_{out} and positive operators R_B acting on $\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}$, which is given by the following expressions

$$R_B = B \otimes I(|\Omega\rangle\langle\Omega|), \quad B(\rho) = \text{Tr}_{\text{in}}[(I_{\text{out}} \otimes \rho^\tau) R_B], \quad (2)$$

where $|\Omega\rangle = \sum_{n=0}^{\infty} |\psi_n\rangle|\psi_n\rangle$ is a maximally entangled vector of $\mathcal{H}_{\text{in}}^{\otimes 2}$ and X^τ denotes transposition of X in the basis $|\psi_n\rangle$. In terms of the operator R_B the covariance property (1) can be written as

$$[R_B, D(\alpha)^{\otimes M} \otimes D(\alpha^*)^{\otimes N}] = 0, \quad \forall \alpha \in \mathbb{C}. \quad (3)$$

In order to deal with this constraint, we introduce the multisplitter operators U_a and U_b that perform the unitary transformations

$$U_a a_k U_a^\dagger = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} e^{\frac{2\pi i k l}{N}} a_l, \quad U_b b_k U_b^\dagger = \frac{1}{\sqrt{M}} \sum_{l=0}^{M-1} e^{\frac{2\pi i k l}{M}} b_l. \quad (4)$$

Notice that such transformations perform a Fourier transform over all input and output modes. Moreover, we will make use of the squeezing transformation $S_{a_0 b_0}$ defined as follows

$$\begin{aligned} [S_{a_0 b_0}, a_n] &= [S_{a_0 b_0}, b_n] = 0, & n > 0, & \quad S_{a_0 b_0} a_0^\dagger S_{a_0 b_0}^\dagger = \mu a_0^\dagger - \nu b_0, \\ S_{a_0 b_0} b_0 S_{a_0 b_0}^\dagger &= \mu b_0 - \nu a_0^\dagger, \end{aligned} \quad (5)$$

with $\mu = \sqrt{M/(M-N)}$ and $\nu = \sqrt{N/(M-N)}$. The squeezing transformation here acts just as an hyperbolic transformation for just modes a_0 and b_0 , by leaving all other modes unaffected. In terms of such operators, condition (3) becomes

$$[S_{a_0 b_0}^\dagger (U_b^\dagger \otimes U_a^\dagger) R_B (U_b \otimes U_a) S_{a_0 b_0}, D_{b_0}(\sqrt{M-N}\alpha)] = 0. \quad (6)$$

Hence, upon introducing an operator B on modes $b_1, \dots, b_{M-1}, a_0, \dots, a_{N-1}$, the operator R_B can be written in the form

$$R_B = (U_b \otimes U_a) S_{a_0 b_0} (I_{b_0} \otimes B) S_{a_0 b_0}^\dagger (U_b^\dagger \otimes U_a^\dagger). \quad (7)$$

Notice that $R_B \geq 0$ is equivalent to $B \geq 0$. The further condition that B is trace-preserving in terms of R_B becomes $\text{Tr}_b[R_B] = I_a$, b and a collectively denoting all output and input modes,

respectively. From the trace and completeness relations for the set of displacement operators, namely $\int d^2\alpha D(\alpha)AD^\dagger(\alpha) = \text{Tr}[A]I$ and $A = \int d^2\alpha \text{Tr}[D^\dagger(\alpha)A]D(\alpha)$, (see, e.g., [13]), the condition $\text{Tr}_b[R_B] = I_a$ is verified if

$$\left(\prod_{i=0}^{M-1} \int d^2\beta_i \right) \left(\bigotimes_{i=0}^{M-1} D_{b_i}(\beta_i) \right) S_{a_0b_0}(I_{b_0} \otimes B) S_{a_0b_0}^\dagger \left(\bigotimes_{i=0}^{M-1} D_{b_i}^\dagger(\beta_i) \right) = I. \quad (8)$$

From the relation $D_{b_0}(\beta_0)S_{a_0b_0} = S_{a_0b_0}D_{b_0}(\mu\beta_0)D_{a_0}^\dagger(\nu\beta_0)$, one obtains the condition

$$\text{Tr}_{b/b_0, a_0}[B] = \nu^2 I_{a/a_0}, \quad (9)$$

where a/a_i denotes all the input modes apart from a_i , and similarly for b/b_i .

We will now consider the map corresponding to

$$B = \nu^2 |0\rangle\langle 0|_{b/b_0} \otimes |0\rangle\langle 0|_{a_0} \otimes I_{a/a_0}. \quad (10)$$

Applying the corresponding map B to a generic N -partite state Ξ we get

$$B(\Xi) = \text{Tr}_a[(I_b \otimes \Xi^\tau)(U_b \otimes U_a)S_{a_0b_0}(I_{b_0} \otimes B)S_{a_0b_0}^\dagger(U_b^\dagger \otimes U_a^\dagger)], \quad (11)$$

which is equivalent to

$$B(\Xi) = \text{Tr}_a[(I_b \otimes U_a^\dagger \Xi^\tau U_a)(U_b \otimes I_a)S_{a_0b_0}(I_{b_0} \otimes B)S_{a_0b_0}^\dagger(U_b^\dagger \otimes I_a)]. \quad (12)$$

Using the expression in equation (10) we obtain

$$B(\Xi) = U_b \left\{ \text{Tr}_{a_0}[(I_{b_0} \otimes \xi_{a_0}^\tau)S_{a_0b_0}(I_{b_0} \otimes |0\rangle\langle 0|_{a_0})S_{a_0b_0}^\dagger] \otimes |0\rangle\langle 0|_{b/b_0} \right\} U_b^\dagger, \quad (13)$$

where $\xi^\tau = \text{Tr}_{a/a_0}[U_a^\dagger \Xi^\tau U_a]$. Notice that

$$\xi = \int \frac{d^2\gamma}{\pi} D(\gamma)^\tau \text{Tr}[(D_{a_0}(\gamma)^\dagger \otimes I_{a/a_0})U_a^\dagger \Xi^\tau U_a] = \int \frac{d^2\gamma}{\pi} D(\gamma)^\tau \text{Tr}[U_a^*(D_{a_0}(\gamma)^* \otimes I_{a/a_0})U_a^\tau \Xi], \quad (14)$$

and taking the complex conjugate of equation (4) we have

$$\begin{aligned} \xi &= \int \frac{d^2\gamma}{\pi} D(\gamma)^\tau \text{Tr}[D(\gamma^*/\sqrt{N})^{\otimes N} \Xi] \\ &= \int \frac{d^2\gamma}{\pi} D(\gamma)^\tau \text{Tr}[(D_{a_0}(\gamma)^* \otimes I_{a/a_0})U_a^\dagger \Xi U_a] = \text{Tr}_{a/a_0}[U_a^\dagger \Xi U_a]. \end{aligned} \quad (15)$$

Now, we can easily evaluate $S_{a_0b_0}(I_{b_0} \otimes |0\rangle\langle 0|_{a_0})S_{a_0b_0}^\dagger$, by expanding the vacuum state as

$$|0\rangle\langle 0|_{a_0} = \int \frac{d^2\gamma}{\pi} e^{-\frac{|\gamma|^2}{2}} D_{a_0}(\gamma), \quad (16)$$

obtaining

$$S_{a_0 b_0}(I_{b_0} \otimes |0\rangle\langle 0|_{a_0})S_{a_0 b_0}^\dagger = \int \frac{v^2 d^2\gamma}{\pi} e^{-\frac{|\gamma|^2}{2}} D_{b_0}(v\gamma^*) \otimes D_{a_0}(\mu\gamma). \quad (17)$$

Hence, equation (13) can be rewritten as

$$B(\Xi) = \int \frac{d^2\gamma}{\pi} U_b(D_{b_0}(\gamma^*) \otimes |0\rangle\langle 0|_{b/b_0})U_b^\dagger e^{-\frac{|\gamma|^2}{2v^2}} \text{Tr}[D_{a_0}(\mu\gamma/v)\xi^\tau]. \quad (18)$$

As an example, we will now consider N displaced thermal states

$$\rho_\alpha \doteq \frac{1}{\bar{n}+1} D(\alpha) \left(\frac{\bar{n}}{\bar{n}+1} \right)^{a^\dagger a} D(\alpha)^\dagger, \quad (19)$$

from which we want to obtain M states, the purest possible. Thanks to the covariance property, it is sufficient to focus attention on the output of $\rho_0^{\otimes N}$. For a tensor product of thermal input states $\Xi = \rho_0^{\otimes N}$, exploiting the fact that $U_a^\dagger(\sum_{j=0}^{N-1} a_j^\dagger a_j)U_a = \sum_{j=0}^{N-1} a_j^\dagger a_j$, we have

$$\xi = \xi^\tau = \rho_0, \quad (20)$$

and recalling the following expression for the thermal states

$$\frac{1}{\bar{n}+1} \left(\frac{\bar{n}}{\bar{n}+1} \right)^{a^\dagger a} = \int \frac{d^2\beta}{\pi} e^{-\frac{|\beta|^2}{2}(2\bar{n}+1)} D(\beta), \quad (21)$$

we obtain

$$\begin{aligned} B(\rho_0^{\otimes N}) &= \int \frac{d^2\gamma}{\pi} U_b(D_{b_0}(-\gamma^*) \otimes |0\rangle\langle 0|_{b/b_0})U_b^\dagger e^{-\frac{|\gamma|^2}{2v^2} [\mu^2(2\bar{n}+1)+1]} \\ &= \int \frac{d^2\gamma}{\pi \bar{n}'} U_b(|\gamma\rangle\langle \gamma|_{b_0} \otimes |0\rangle\langle 0|_{b/b_0})U_b^\dagger e^{-\frac{|\gamma|^2}{\bar{n}'}} \\ &= \int \frac{d^2\gamma}{\pi \bar{n}'} \left| \gamma/\sqrt{M} \right\rangle \left\langle \gamma/\sqrt{M} \right|^{\otimes M} e^{-\frac{|\gamma|^2}{2\bar{n}'}} = \int \frac{M d^2\gamma}{\pi \bar{n}'} |\gamma\rangle\langle \gamma|^{\otimes M} e^{-\frac{M|\gamma|^2}{\bar{n}'}} \end{aligned} \quad (22)$$

where

$$2\bar{n}' + 1 = \frac{1}{v^2} [\mu^2(2\bar{n}+1) + 1] = \frac{2M\bar{n} + 2M - N}{N}. \quad (23)$$

The above state is permutation-invariant and separable, with thermal local state at each mode with average thermal photon number

$$\bar{n}'' = \frac{\bar{n}'}{M} = \frac{M\bar{n} + M - N}{MN}. \quad (24)$$

More generally, for any state Ξ , the choice (10) gives M identical clones whose state can be written as

$$\rho' = \int \frac{d^2\alpha}{\pi} e^{-\frac{|\alpha|^2}{2}(\frac{1}{N} - \frac{2}{M} + 1)} \{\text{Tr}[\Xi D^\dagger(\alpha/N)^{\otimes N}]\} D(\alpha). \quad (25)$$

Since for any mode c one has

$$\Delta x_c^2 + \Delta y_c^2 = \frac{1}{2} + \langle c^\dagger c \rangle - |\langle c \rangle|^2, \quad (26)$$

it is easy to verify that the superbroadcasting condition (output total noise in conjugate quadratures smaller than the input one), is equivalent to require smaller photon number at the output than at the input, namely

$$\bar{n} \geq \frac{M\bar{n} + M - N}{MN} \Leftrightarrow \bar{n} \geq \frac{M - N}{M(N - 1)}. \quad (27)$$

This can be true for any $N > 1$ and to any $M \leq \infty$, since

$$\lim_{M \rightarrow \infty} \frac{M - N}{M(N - 1)} = \frac{1}{N - 1} > 0. \quad (28)$$

3. Proof of optimality for the channel in equation (11)

Actually, the solution given in equation (24) is optimal. To prove this, in the following we will show that the expectation of the total number of photons $\text{Tr}[\sum_{l=0}^{M-1} b_l^\dagger b_l B(\rho_0^{\otimes N})]$ of the M clones of ρ cannot be smaller than $M\bar{n}''$. Since the multisplitter preserves the total number of photons we have to consider the trace

$$W \doteq \text{Tr} \left[\left(\sum_{l=0}^{M-1} b_l^\dagger b_l \otimes (U_a^\dagger \rho_0^{\tau \otimes N} U_a) \right) S_{a_0 b_0} (I_{b_0} \otimes B) S_{a_0 b_0}^\dagger \right]. \quad (29)$$

We can write $W = W_0 + \sum_{l=1}^{M-1} W_l$, with

$$\begin{aligned} W_0 &\doteq \text{Tr} \left[S_{a_0 b_0}^\dagger \left((b_0^\dagger b_0 \otimes I_{b/b_0}) \otimes (U_a^\dagger \rho_0^{\tau \otimes N} U_a) \right) S_{a_0 b_0} (I_{b_0} \otimes B) \right], \\ W_l &\doteq \text{Tr} \left[S_{a_0 b_0}^\dagger \left((I_{b/b_l} \otimes b_l^\dagger b_l) \otimes (U_a^\dagger \rho_0^{\tau \otimes N} U_a) \right) S_{a_0 b_0} (I_{b_0} \otimes B) \right], \end{aligned} \quad (30)$$

for $1 \leq l \leq M - 1$. Now, since $W_l \geq 0$, $W \geq W_0$. Moreover, using the identity $c^\dagger c = -\partial_{\alpha\alpha^*} e^{|\alpha|^2/2} D_c(\alpha)|_{\alpha=\alpha^*=0}$, one obtains

$$\begin{aligned} &\text{Tr}_{b_0} [S_{a_0 b_0}^\dagger (b_0^\dagger b_0 \otimes \sigma) S_{a_0 b_0}] \\ &= -\partial_{\alpha\alpha^*} \int \frac{d^2\gamma}{\pi} \text{Tr}_{b_0} [D_{b_0}(\mu\alpha - \nu\gamma^*) \otimes D_{a_0}(\mu\gamma - \nu\alpha^*)] \text{Tr}[D(\gamma)^\dagger \sigma] e^{\frac{|\alpha|^2}{2}} \Big|_{\alpha=\alpha^*=0} \\ &= -\frac{1}{\nu^2} \partial_{\alpha\alpha^*} e^{-\frac{|\alpha|^2}{\nu^2}} e^{\frac{\alpha^*}{\nu} a_0^\dagger} e^{-\frac{\alpha}{\nu} a_0} \text{Tr} \left[e^{\frac{\mu\alpha^*}{\nu} a_0^\dagger} e^{-\frac{\mu\alpha}{\nu} a_0} \sigma \right] \Big|_{\alpha=\alpha^*=0} = \frac{a_0^\dagger a_0 + \mu^2 \text{Tr}[a_0^\dagger a_0 \sigma] + 1}{\nu^4}, \end{aligned} \quad (31)$$

then, from equation (9) and positivity of B , one has

$$W_0 = \frac{\text{Tr}[(I_{b/b_0} \otimes a_0^\dagger a_0 \otimes I_{a/a_0})\{I_{b/b_0} \otimes (U_a \rho_0^{\otimes N} U_a^\dagger)^\tau\} B]}{v^4} + \frac{\mu^2 \text{Tr}[(I_{b/b_0} \otimes I_{a_0} \otimes \text{Tr}_{a_0}[a_0^\dagger a_0 (U_a \rho_0^{\otimes N} U_a^\dagger)^\tau]) B] + v^2}{v^4} \geq \frac{\mu^2 \bar{n} + 1}{v^2} = \frac{N}{M} \bar{n} + \frac{M - N}{N} = M \bar{n}''. \quad (32)$$

In fact, one can easily check that the choice of B in equation (10) saturates the bound (32).

Also the more general solution given in equation (25) is optimal, in the sense that it represents the state of M identical clones with minimal photon number, which is given by

$$\text{Tr}[b^\dagger b \rho'] = \frac{\text{Tr}[a^\dagger a \rho_0]}{N} + \frac{1}{N} - \frac{1}{M}. \quad (33)$$

Notice that for $\bar{n} = 0$, one has N coherent states at the input, and $\bar{n}'' = (M - N)/MN$, namely one recovers the optimal cloning for coherent states of [14].

From equation (26), one can see that our optimization maximally reduces the total noise in conjugate quadratures. Alternatively, one might minimize the output entropy, which would be informationally more satisfactory. This case, however, turns out to be a nontrivial task, and is beyond the scope of this paper.

4. Purification

For $M < N$, one can look for the optimal ‘purification’ map with M output systems. The result can be obtained as in section 2, provided that we replace the operator $S_{a_0 b_0}$ in equation (5) with

$$[T_{a_0 b_0}, a_n] = [T_{a_0 b_0}, b_n] = 0, \quad n > 0, \quad T_{a_0 b_0} a_0 T_{a_0 b_0}^\dagger = \mu a_0 - \nu b_0^\dagger, \quad T_{a_0 b_0} b_0^\dagger T_{a_0 b_0}^\dagger = \mu b_0^\dagger - \nu a_0, \quad (34)$$

where now $\mu = \sqrt{N/(N - M)}$ and $\nu = \sqrt{M/(N - M)}$, and the constraint in equation (6) with

$$[T_{a_0 b_0}^\dagger (U_b^\dagger \otimes U_a^\dagger) R_B (U_b \otimes U_a) T_{a_0 b_0}, D_{a_0}(\sqrt{N - M} \alpha)] = 0, \quad (35)$$

for all α . Consequently, R_B has the form

$$R_B = (U_b \otimes U_a) T_{a_0 b_0} (I_{a_0} \otimes B) T_{a_0 b_0}^\dagger (U_b^\dagger \otimes U_a^\dagger), \quad (36)$$

and trace preservation is equivalent to

$$\text{Tr}_b[B] = \mu^2 I_{a/a_0}. \quad (37)$$

Now, we consider the map with

$$B = \mu^2 |0\rangle\langle 0|_b \otimes I_{a/a_0}. \quad (38)$$

The corresponding output for given input state Ξ is given by

$$B(\Xi) = \int \frac{d^2\gamma}{\pi} U_b(D_{b_0}(\gamma) \otimes |0\rangle\langle 0|_{b/b_0}) U_b^\dagger e^{-\frac{|\gamma|^2}{2\mu^2}} \text{Tr}[D_{a_0}(-\nu\gamma^*/\mu)\xi^\tau], \quad (39)$$

where $\xi = \text{Tr}_{a/a_0}[U_a \Xi U_a^\dagger]$. For $\Xi = \rho_0^{\otimes N}$, we have $\xi = \rho_0$ and

$$B(\rho_0^{\otimes N}) = \int \frac{d^2\gamma}{\pi} e^{-\frac{|\gamma|^2}{2\mu^2}[\nu^2(2\bar{n}+1)+1]} U_b(D_{b_0}(\gamma) \otimes |0\rangle\langle 0|_{b/b_0}) U_b^\dagger. \quad (40)$$

The integral gives a thermal state for the mode b_0 with average photon number \bar{n}' such that $2\bar{n}' + 1 = [\nu^2(2\bar{n} + 1) + 1]/\mu^2 = 2(M/N)\bar{n} + 1$, namely $\bar{n}' = (M/N)\bar{n}$. Finally, one has

$$B(\rho_0^{\otimes N}) = \int \frac{M d^2\gamma}{\bar{n}'\pi} e^{-\frac{M|\gamma|^2}{\bar{n}'}} |\gamma\rangle\langle \gamma|^{\otimes M}. \quad (41)$$

Hence, the single-site reduced state is a thermal state with a number of thermal photons

$$\bar{n}'' = \frac{\bar{n}}{N}, \quad (42)$$

which is rescaled with respect to the input by a factor N , independently of the number of output copies. The same analysis as in section 3 shows that this is the minimum output number compatible with complete positivity of the map B , and then is optimal.

For a generic input state Ξ the local output state is given by

$$\rho' = \int \frac{d^2\alpha}{\pi} e^{-\frac{|\alpha|^2}{2}(1-\frac{1}{N})} \{\text{Tr}[\Xi D^\dagger(\alpha/N)^{\otimes N}]\} D(\alpha). \quad (43)$$

Notice that both equations (23) and (42) give $\bar{n}'' = \frac{\bar{n}}{N}$ also for $M = N$, and this result can be proved as follows. The difference from the previous proof resides in the fact that the squeezing operator $S_{a_0b_0}$ is ill defined in this case. However, once we unitarily transform $D(\alpha)^{\otimes N} \otimes D(\alpha)^{* \otimes N}$ to $D_{b_0}(\sqrt{N}\alpha) \otimes I_{b/b_0} \otimes D(\sqrt{N}\alpha)_{a_0}^* \otimes I_{a/a_0}$, the squeezing operator on modes a_0 and b_0 is not needed, and it is sufficient to remark that the representation $D_{b_0}(\sqrt{N}\alpha) \otimes D_{a_0}(\sqrt{N}\alpha)^*$ is abelian, and its joint eigenvectors can be written as

$$|D(\beta)\rangle\rangle \doteq \sum_{m,n=0}^{\infty} \langle m|D(\beta)|n\rangle |m\rangle_{b_0} |n\rangle_{a_0}. \quad (44)$$

Consequently, the covariance condition for the map B is given by

$$R_B = (U_b \otimes U_a) \int \frac{d^2\gamma}{\pi} |D(\gamma)\rangle\rangle \langle\langle D(\gamma)|_{a_0b_0} \otimes \Delta_{a/a_0,b/b_0}(\gamma)(U_b^\dagger \otimes U_a^\dagger), \quad (45)$$

with the trace-preserving constraint expressed by

$$\int \frac{d^2\gamma}{\pi} \text{Tr}_{b/b_0}[\Delta_{a/a_0,b/b_0}(\gamma)] = I_{a/a_0}. \quad (46)$$

We consider the following form for $\Delta_{a/a_0, b/b_0}(\gamma)$

$$\Delta_{a/a_0, b/b_0}(\gamma) = \pi \delta^2(\gamma) I_{a/a_0} \otimes |0\rangle\langle 0|_{b/b_0}, \quad (47)$$

which gives

$$R_B = (U_b \otimes U_a) |I\rangle\langle I|_{a_0 b_0} \otimes |0\rangle\langle 0|_{b/b_0} \otimes I_{a/a_0} (U_b \otimes U_a)^\dagger, \quad (48)$$

and then we can prove optimality by the same technique used in the other cases. The output of Ξ is given by

$$B(\Xi) = U_b (\xi_{b_0}^\dagger \otimes |0\rangle\langle 0|_{b/b_0}) U_b^\dagger, \quad (49)$$

where $\xi = \text{Tr}_{a/a_0} [U_a^\dagger \Xi U_a]$, and for thermal states $\Xi = \rho_0^{\otimes N}$ we have $\xi = \rho_0$ and

$$B(\rho_0^{\otimes N}) = \int \frac{N d^2 \gamma}{\pi \bar{n}} e^{-\frac{N |\gamma|^2}{\bar{n}}} |\gamma\rangle\langle \gamma|^{\otimes N}, \quad (50)$$

which is separable, and its local states are thermal states with

$$\bar{n}'' = \frac{\bar{n}}{N}. \quad (51)$$

5. Phase-conjugating broadcasting

We now consider the problem of broadcasting with simultaneous phase-conjugate output. This means that we look for the optimal transformation where the average of the output field of each copy is the complex conjugate with respect to the value of the input one. The covariance property of such a map is the following

$$C(D(\alpha)^{\otimes N} \Xi D(\alpha)^{\dagger \otimes N}) = D(\alpha)^{* \otimes M} C(\Xi) D(\alpha)^{T \otimes M}, \quad (52)$$

for all α , and in terms of R_C this corresponds to

$$[D(\alpha)^{* \otimes (M+N)}, R_C] = 0. \quad (53)$$

We will use the same multisplitters defined in equation (4), and introduce the following beam-splitter

$$\begin{aligned} [U_{a_0 b_0}, a_n] = [U_{a_0 b_0}, b_n] = 0, \quad n > 0, \quad U_{a_0 b_0} b_0 U_{a_0 b_0}^\dagger = \eta b_0 + \theta a_0, \\ U_{a_0 b_0} a_0 U_{a_0 b_0}^\dagger = -\theta b_0 + \eta a_0, \end{aligned} \quad (54)$$

with $\eta = \sqrt{\frac{M}{M+N}}$ and $\theta = \sqrt{\frac{N}{M+N}}$. The covariance relation in equation (53) can be written

$$[U_{a_0 b_0}^\dagger (U_b^\dagger \otimes U_a^\dagger) R_C (U_b \otimes U_a) U_{a_0 b_0}, D_{b_0}(\sqrt{M+N} \alpha)^*] = 0. \quad (55)$$

Analogously to the previous sections, the covariance condition translates in the following form for R_C :

$$R_C = U_b U_a U_{a_0 b_0} (I_{b_0} \otimes C) U_{a_0 b_0}^\dagger U_b^\dagger U_a^\dagger, \quad (56)$$

where C is an operator on modes $b_1, \dots, b_{M-1}, a_0, \dots, a_{N-1}$, and the trace-preserving condition requires that

$$\left(\prod_{i=0}^{M-1} \int d^2 \beta_i \right) \left(\bigotimes_{i=1}^{M-1} D_{b_i}(\beta_i) \right) U_{a_0 b_0} (I_{b_0} \otimes C) U_{a_0 b_0}^\dagger \left(\bigotimes_{i=1}^{M-1} D_{b_i}^\dagger(\beta_i) \right) = I, \quad (57)$$

which finally gives

$$\text{Tr}_{b/b_0, a_0}[C] = \theta^2 I_{a/a_0}. \quad (58)$$

We now consider the map corresponding to

$$C = \theta^2 |0\rangle\langle 0|_{b/b_0} \otimes |0\rangle\langle 0|_{a_0} \otimes I_{a/a_0}. \quad (59)$$

Applying such a map to a generic N -partite state Ξ we get

$$\begin{aligned} C(\Xi) &= U_b \text{Tr}_a [(I_b \otimes U_a^\dagger \Xi^\tau U_a) U_{a_0 b_0} (I_{b_0} \otimes C) U_{a_0 b_0}^\dagger] U_b^\dagger \\ &= U_b (\text{Tr}_{a_0} [(I_{b_0} \otimes \xi^\tau) U_{a_0 b_0} (I_{b_0} \otimes |0\rangle\langle 0|_{a_0}) U_{a_0 b_0}^\dagger] |0\rangle\langle 0|_{b/b_0}) U_b^\dagger, \end{aligned} \quad (60)$$

where $\xi^\tau = \text{Tr}_{a/a_0} [U_a^\dagger \Xi^\tau U_a]$. Moreover, one has

$$U_{a_0 b_0} (I_{b_0} \otimes |0\rangle\langle 0|_{a_0}) U_{a_0 b_0}^\dagger = \int \frac{\theta^2 d^2 \gamma}{\pi} |\eta \gamma\rangle\langle \eta \gamma|_{b_0} \otimes |\theta \gamma\rangle\langle \theta \gamma|_{a_0}, \quad (61)$$

and equation (60) gives

$$C(\Xi) = U_b (H(\xi) \otimes |0\rangle\langle 0|_{b/b_0}) U_b^\dagger, \quad (62)$$

where $\xi = \text{Tr}_{a/a_0} [U_a \Xi U_a^\dagger]$, and

$$H(\rho) = \int \frac{d^2 \gamma}{\pi} |(\eta/\theta) \gamma\rangle\langle \gamma^* | \xi | \gamma^* \rangle \langle (\eta/\theta) \gamma|. \quad (63)$$

For $\Xi = \rho_0^{\otimes N}$, we have simply $\xi = \rho_0$, and this implies that a simple scheme to achieve this map is the following. Firstly, the N input states interact through an N -splitter, then the system labelled 0 carrying all the information about the coherent signal is measured by heterodyne detection, and for any outcome γ a coherent state with amplitude $\sqrt{(M/N)} \gamma^*$ is generated. Finally, the prepared state is sent through an M -splitter along with $M - 1$ modes in the vacuum state.

The output state $C(\rho_0^{\otimes N})$ is now given by

$$C(\rho_0^{\otimes N}) = \frac{1}{\bar{n} + 1} \int \frac{d^2 \gamma}{\pi} e^{-\frac{|\gamma|^2}{\bar{n} + 1}} U_b (|\sqrt{M/N} \gamma\rangle\langle \sqrt{M/N} \gamma| \otimes |0\rangle\langle 0|_{b/b_0}) U_b^\dagger, \quad (64)$$

which is equal to

$$C(\rho_0^{\otimes N}) = \frac{N}{\bar{n} + 1} \int \frac{d^2\gamma}{\pi} e^{-\frac{N|\gamma|^2}{\bar{n}+1}} |\gamma\rangle\langle\gamma|, \quad (65)$$

and its single-site reduced state is simply a thermal state with

$$\bar{n}'' = \frac{\bar{n} + 1}{N}. \quad (66)$$

Notice that this is independent of the number of output copies and is the same average number as the one for superbroadcasting in the limit $M \rightarrow \infty$. More generally, the local output for generic input state Ξ is

$$\rho' = \int \frac{d^2\alpha}{\pi} e^{-\frac{|\alpha|^2}{2}(1+\frac{1}{N})} \{\text{Tr}[\Xi D^\dagger(\alpha/N)^{\otimes N}]\} D(\alpha). \quad (67)$$

The proof of optimality is analogous to the proof for the superbroadcasting map. It is sufficient to replace $U_{a_0b_0}$ with $S_{a_0b_0}$ in equations (29) and (30).

6. A proof of the optimality in terms of linear amplifiers

We are interested in a transformation that provides M (generally correlated) modes b_0, b_1, \dots, b_{M-1} from N uncorrelated modes a_0, a_1, \dots, a_{N-1} , such that the unknown complex amplitude is preserved and the output has minimal phase-insensitive noise. In formula, we have input uncorrelated modes

$$\langle a_i \rangle = \alpha, \quad \Delta x_{a_i}^2 + \Delta y_{a_i}^2 = \gamma_i \geq \frac{1}{2}, \quad (68)$$

for all $i = 0, 1, \dots, N - 1$, where Heisenberg uncertainty relation is taken into account. The output modes should satisfy

$$\langle b_i \rangle = \alpha, \quad \Delta x_{b_i}^2 + \Delta y_{b_i}^2 = \Gamma \geq \frac{1}{2}, \quad (69)$$

and we look for the minimal Γ . The minimal Γ can be obtained by applying a fundamental theorem for phase-insensitive linear amplifiers [15]: the sum of the uncertainties of conjugated quadratures of a phase-insensitive amplified mode with (power) gain G is bounded as follows.

$$\Delta X_B^2 + \Delta Y_B^2 \geq G(\Delta X_A^2 + \Delta Y_A^2) + \frac{G-1}{2}, \quad (70)$$

where A and B denote the input and the amplified mode, respectively. Our transformation can be seen as a phase-insensitive amplification from the mode $A = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} a_i$ to the mode $B = \frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} b_i$ with gain $G = M/N$, and hence equation (70) should hold. Notice that generally for any mode c one has

$$\Delta x_c^2 + \Delta y_c^2 = \frac{1}{2} + \langle c^\dagger c \rangle - |\langle c \rangle|^2. \quad (71)$$

Hence, the bound can be rewritten as

$$\langle B^\dagger B \rangle - |\langle B \rangle|^2 \geq G(\langle A^\dagger A \rangle + 1 - |\langle A \rangle|^2) - 1. \quad (72)$$

In the present case, since modes a_i are uncorrelated, one has

$$\begin{aligned} \langle A^\dagger A \rangle &= \frac{1}{N} \sum_{i,j=0}^{N-1} \langle a_i^\dagger a_j \rangle = \frac{1}{N} \left(\sum_{i=0}^{N-1} \langle a_i^\dagger a_i \rangle + \sum_{i \neq j} \langle a_i^\dagger a_j \rangle \right) = (\gamma + |\alpha|^2 - \frac{1}{2}) + (N-1)|\alpha|^2 \\ &= \gamma + N|\alpha|^2 - \frac{1}{2}, \end{aligned} \quad (73)$$

where $\gamma = \frac{1}{N} \sum_{i=0}^{N-1} \gamma_i$, and so the bound equation (71) is written as

$$\langle B^\dagger B \rangle \geq G(\gamma + \frac{1}{2}) - 1 + M|\alpha|^2. \quad (74)$$

On the other hand, one has

$$\langle B^\dagger B \rangle = \frac{1}{M} \sum_{i,j=0}^{M-1} \langle b_i^\dagger b_j \rangle \leq \frac{1}{M} \sum_{i,j=0}^{M-1} \sqrt{\langle b_i^\dagger b_i \rangle \langle b_j^\dagger b_j \rangle} = M(\Gamma + |\alpha|^2 - \frac{1}{2}). \quad (75)$$

Equations (74) and (75) together give the bound for the minimal noise Γ

$$\Gamma - \frac{1}{2} \geq \frac{1}{N}(\gamma - \frac{1}{2}) + \frac{1}{N} - \frac{1}{M}. \quad (76)$$

The example in the previous sections corresponds to $\gamma = \bar{n} + \frac{1}{2}$ and $\Gamma = \bar{n}'' + \frac{1}{2}$. A similar derivation gives a bound for purification, where $N > M$. In such a case $G < 1$, and equation (70) is replaced with

$$\Delta X_B^2 + \Delta Y_B^2 \geq G(\Delta X_A^2 + \Delta Y_A^2) + \frac{1-G}{2}, \quad (77)$$

and one obtains the bound

$$\Gamma - \frac{1}{2} \geq \frac{1}{N}(\gamma - \frac{1}{2}). \quad (78)$$

We would like to stress that the derivation of all bounds in the present section relies on the theorem of the added noise in *linear* amplifiers, namely only linear transformations of modes are considered. Hence, in principle, these bounds might be violated by more exotic and nonlinear transformations. Therefore, the derivation of equation (32) is stronger, since it has general validity.

By a similar derivation, using the bound for phase-conjugated amplifiers $\Delta X_B^2 + \Delta Y_B^2 \geq G(\Delta X_A^2 + \Delta Y_A^2) + \frac{G-1}{2}$, one can obtain the bound for phase-conjugation broadcasting

$$\Gamma - \frac{1}{2} \geq \frac{1}{N}(\gamma + \frac{1}{2}). \quad (79)$$

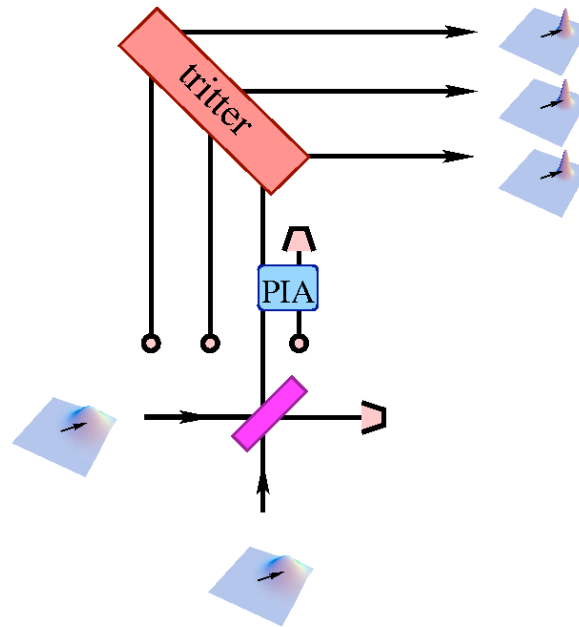


Figure 1. Experimental scheme to achieve optimal superbroadcasting from 2 to 3 copies. This setup involves just a beam splitter, a phase-insensitive amplifier and a tritter, which in turn can be implemented by two suitably balanced beam splitters. The phase-insensitive amplifier can be implemented by a beam splitter and heterodyne-assisted feed-forward. The output copies carrying the same signal as the input ones are locally more pure, the noise being shifted to classical correlations between them.

7. Experimental implementation

The optimal broadcasting can be easily implemented by means of an inverse N -splitter which concentrates the signal in one mode and discards the other $N - 1$ modes. The mode is then amplified by a phase-insensitive amplifier with power gain $G = M/N$. Finally, the amplified mode is distributed by mixing it in an M -splitter with $M - 1$ vacuum modes. Each mode is then found in the state of equation (25). In the concentration stage, the N modes with amplitude $\langle a_i \rangle = \alpha$ and noise $\Delta x_i^2 + \Delta y_i^2 = \gamma_i$ are reduced to a single mode with amplitude $\sqrt{N}\alpha$ and noise γ . The amplification stage gives a mode with amplitude $\sqrt{M}\alpha$ and noise $\gamma' = \gamma \frac{M}{N} + \frac{M}{2N} - \frac{1}{2}$. Finally, the distribution stage gives M modes, with amplitude α and noise $\Gamma = \frac{1}{M} (\gamma' + \frac{M-1}{2})$ each. In figure 1, we sketch the scheme for 2 to 3 superbroadcasting.

In [16], it was shown experimentally that phase-insensitive amplification can be obtained by a setup consisting of a beam-splitter, a heterodyne detector and a conditional displacement. In the following, we give an algebraic derivation of this result. Consider a mode in a state $\rho = \int \frac{d^2\gamma}{\pi} f(\gamma) D(\gamma)$ coupled to another mode in the vacuum through a beam-splitter with transmissivity τ . The output is given by the bipartite state σ

$$\sigma = \int \frac{d^2\beta d^2\gamma}{\pi^2} e^{-\frac{|\tau\beta - \sqrt{1-\tau^2}\gamma|^2}{2}} f(\tau\gamma + \sqrt{1-\tau^2}\beta) D(\gamma) \otimes D(\beta), \quad (80)$$

where we performed the change of variables $\beta \rightarrow \tau\beta + \sqrt{1 - \tau^2}\gamma$, $\gamma \rightarrow \tau\gamma - \sqrt{1 - \tau^2}\beta$. Now, the reflected mode is measured by heterodyne detection, and conditionally on the measurement outcome α , a displacement $D(k\alpha)$ is performed on the transmitted mode, whose state is then given by

$$\begin{aligned}
\rho' &= \int \frac{d^2\alpha d^2\beta d^2\gamma}{\pi^3} e^{-\frac{|\tau\beta - \sqrt{1 - \tau^2}\gamma|^2}{2}} f(\tau\gamma + \sqrt{1 - \tau^2}\beta) D(k\alpha) D(\gamma) D(k\alpha)^\dagger \langle 0 | D(\alpha)^\dagger D(\beta) D(\alpha) | 0 \rangle \\
&= \int \frac{d^2\alpha d^2\beta d^2\gamma}{\pi^3} e^{-\frac{|\tau\beta - \sqrt{1 - \tau^2}\gamma|^2}{2}} f(\tau\gamma + \sqrt{1 - \tau^2}\beta) e^{\alpha(k\gamma^* - \beta^*) - \text{c.c.}} D(\gamma) e^{-\frac{|\beta|^2}{2}} \\
&= \int \frac{d^2\beta d^2\gamma}{\pi^2} \delta^{(2)}(k\gamma - \beta) e^{-\frac{|\tau\beta - \sqrt{1 - \tau^2}\gamma|^2}{2}} f(\tau\gamma + \sqrt{1 - \tau^2}\beta) e^{-\frac{|\beta|^2}{2}} \\
&= \int \frac{d^2\gamma}{\pi} f(\gamma(\tau + k\sqrt{1 - \tau^2})) e^{-\frac{|\gamma|^2}{2} [k^2 + (k\tau - \sqrt{1 - \tau^2})^2]} D(\gamma). \tag{81}
\end{aligned}$$

On the other hand, the action of a phase-insensitive amplifier on ρ can be easily calculated and produces the partial output state

$$\rho'' = \int \frac{d^2\gamma}{\pi} e^{-\frac{|\gamma|^2 v^2}{2}} f(\mu\gamma) D(\gamma). \tag{82}$$

The following conditions

$$\mu = \tau + k\sqrt{1 - \tau^2}, \quad v^2 = k^2 + (k\tau - \sqrt{1 - \tau^2})^2, \quad \mu^2 - v^2 = 1, \tag{83}$$

which are equivalent to

$$k = v, \quad \tau = \frac{1}{\mu}, \tag{84}$$

imply that $\rho' = \rho''$. Hence, by tuning the beam splitter transmissivity and the parameter of the conditional displacement k , one can then simulate the amplifier by a linear device assisted by heterodyne and feed-forward.

The optimal phase-conjugated broadcasting can be obtained by replacing the linear amplifier with a heterodyne measurement and preparation of a coherent state with conjugate phase and amplified intensity. For achieving the optimal purification, one simply uses an inverse N -splitter which concentrates the signal in one mode and discards the other $N - 1$ modes. Then by N -splitting with $N - 1$ vacuum modes, one obtained N purified signals (although classically correlated).

8. Conclusion

In conclusion, we proved that broadcasting of M copies of a mixed radiation state starting from $N < M$ copies is possible, even with lowering the total noise in conjugate quadratures. Since the noise cannot be removed without violating the quantum data processing theorem, the price to pay for having higher purity at the output is that the output copies are correlated. Essentially noise is moved from local states to their correlations, and our superbroadcasting channel does

this optimally. We obtained similar results also for purification (i.e. $M \leq N$), along with the case of simultaneous broadcasting and phase-conjugation, with the output copies carrying a signal which is complex-conjugated of the input one. Despite the role that correlations play in this effect, no entanglement is present in the output (as long as the single input copy has a positive P -function), as it can be seen by the analytical expression of the output states. Moreover, a practical and very simple scheme for experimental achievement of the maps has been shown, involving mainly passive media and only one parametric amplifier. The superbroadcasting effect has a relevance from the fundamental point of view, opening new perspectives in the understanding of correlations and their interplay with noise, but may be also promising from a practical point of view, for communication tasks in the presence of noise.

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