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Universal quantum estimation

Giacomo Mauro D'Ariano

Istituto Nazionale di Fisica della Materia, Sezione di Pavia, via A. Bassi 6, I-27100 Pavia, Italy

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Abstract

A general method is presented for estimating the ensemble average of all operators of arbitrary quantum system from a set of measurements of a *quorum* of observables. A procedure for deconvolving any kind of instrumental noise is established. Physical implementations and measuring apparatuses are considered. Existing measuring procedures are derived as examples of application of the present general method. New measuring procedures are obtained which apply to different physical contexts. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the jargon of quantum mechanics an *observable* corresponds to a selfadjoint operator O acting on the Hilbert space $\mathscr{H}$ of the quantum system $\mathscr{S}$. However, the problem of which operators correspond to actual observables remains unsolved [1]. Moreover, in practical situations one often measures *complex* quantities – e.g. the e.m. field – or physical parameters – e.g. phase shifts – which do not have selfadjoint counterpart. Quantum estimation theory [2] provides a general framework for such kind of measurements, and complex-field measurements – corresponding to jointly measuring the noncommuting position and momentum of a harmonic oscillator [3] – are routine heterodyne detection. In recent years we have witnessed an increasing interest in the possibility of measuring the density matrix of the quantum state itself [4,5], and from the first theoretical formulation [6] this chance has finally entered the realm of experiments [7,8]. Homodyne tomography (see Ref. [9][1]) is becoming standard in optical labs [12]: this method measures the matrix elements of the e.m. field state by averaging special functions of the field quadratures – the analogue of all linear combinations of position and momentum of a harmonic oscillator. The method has been extended to

E-mail address: dariano@pv.infn.it (G.M. D'Ariano).

[1] The first exact technique to estimate number-state matrix elements by averaging functions of homodyne data was presented in Ref. [10]; for a review see Ref. [11].

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the estimation of the ensemble average $\langle A \rangle$ of any field operator A [13][2].

The set of quadratures in homodyne tomography is an example of *quorum* [6] of observables, namely a 'complete' set of noncommuting observables for determining the quantum state of the system. In this letter the concept of quorum will be the basis of a simple general method for estimating the ensemble average of all operators of arbitrary quantum system. As we will see in the following, the method provides a concrete framework to design measuring apparatuses for estimation, taking into account also instrumental noise of any kind in the measurement. The method will be presented at an intermediate level of generality, in order to keep this letter simply readable: routes to generalizations will be given at the end.

2. The estimation rule

I will call the set $\mathscr{Q}=\{Q_\lambda\}$ of observables Q_λ, $\lambda\in\boldsymbol{\Lambda}$, a *quorum* for $\mathscr{S}$ if it is possible to estimate the ensemble average $\langle A \rangle$ of any linear operator $A\in\mathscr{L}(\mathscr{H})$ by using only measurement outcomes of quorum observables. An *unbiased estimation rule* $\mathscr{E}$ for the quorum $\mathscr{Q}$ assigns to every operator A an operator valued function of $Q_\lambda\in\mathscr{Q}$[3], the *unbiased estimator* $\mathscr{E}_A(Q_\lambda)$, such that the ensemble average $\langle A \rangle = \mathrm{Tr}[A\varrho]$ for arbitrary *unknown* state ϱ can be obtained by averaging over the quorum as follows

$$\langle A \rangle = \int_{\boldsymbol{\Lambda}} \mathrm{d}\mu(\lambda)\, \langle \mathscr{E}_A(Q_\lambda)\rangle\,, \tag{1}$$

where μ is a probability measure over $\boldsymbol{\Lambda}$, and the integral is a sum for discrete set $\boldsymbol{\Lambda}$ (the explicit dependence of $\mathscr{E}_A$ on λ is omitted for simplicity of notation).

Eq. (1) corresponds to the following *estimation procedure* for $\langle A \rangle$: i) select an observable Q_λ randomly in the quorum $\mathscr{Q}$ according to the probability measure μ; ii) measure Q_λ and evaluate the function $\mathscr{E}_A$ of the outcome; iii) average the result over many measurements with different $Q_\lambda\in\mathscr{Q}$, achieving the expectation $\langle A \rangle$ in the limit of infinitely many measurements. Notice that the ensemble average $\langle A \rangle$ of any operator $A\in\mathscr{L}(\mathscr{H})$ is obtained from the *same set of data* using the same fixed estimation rule.

Eq. (1) must be true for arbitrary ϱ, whence

$$A = \int_{\boldsymbol{\Lambda}} \mathrm{d}\mu(\lambda)\, \mathscr{E}_A(Q_\lambda)\,, \tag{2}$$

with integral converging for expected values as in Eq. (1). Notice that the estimation rule is generally not unique, since there exist *null estimators* $\mathscr{N}$ over $\mathscr{Q}$ satisfying the identity

$$\int_{\boldsymbol{\Lambda}} \mathrm{d}\mu(\lambda)\, \mathscr{N}(Q_\lambda) = 0\,. \tag{3}$$

The existence of null estimators sets an equivalence relation $\simeq$ between unbiased estimators (two estimators are equivalent if they differ by a null estimator).

We now derive a general estimation rule abstractly: physical implementations and apparatuses will be considered later. An unbiased estimation rule is obtained from any (Lie) group $\boldsymbol{T}$ of transformations $g\in\boldsymbol{T}$ with invariant measure $\mathrm{d}g$ and unitary irreducible representation[4] (UIR) R over the Hilbert space $\mathscr{H}$ of the quantum system (for simplicity we consider $\boldsymbol{T}$ unimodular and R square integrable). The following selfadjoint involution $E=E^\dagger\equiv E^{-1}$ on $\mathscr{H}\otimes\mathscr{H}$

$$E = \int_{\boldsymbol{T}} \mathrm{d}g\, R^\dagger(g)\otimes R(g)\,, \tag{4}$$

is an *intertwining operator*, namely, for any two operators A and B one has the identity

$$E(A\otimes B) = (B\otimes A)\,E\,, \tag{5}$$

which is easily proved by the first Schur lemma [15]. The invariant measure $\mathrm{d}g$ can be normalized as

[2] For infinite dimensional $\mathscr{H}$ one cannot estimate the ensemble average $\langle A \rangle = \mathrm{Tr}[A\varrho]\equiv\sum_{nm=1}^{\infty}\varrho_{nm}A_{mn}$ from the measured matrix elements ϱ_{nm} of the density operator, because the statistical error for matrix elements ϱ_{nm} is typically non vanishing for $n,m\to\infty$, and the trace sum for $\langle A \rangle$ is affected by unbounded statistical error if A_{nm} is not vanishing sufficiently fast for $n,m\to\infty$. This point is well clarified in the case of homodyne tomography in Ref. [14]

[3] $\mathscr{E}_A(Q_\lambda)$ is a function of Q_λ in the sense that it shares the same spectral decomposition of Q_λ.

[4] For a simple and self-contained book on grouprepresentation theory for physicists, the reader is addressed to Ref. [15].

$\int_{\boldsymbol{T}} \mathrm{d}g\, |\langle u|R(g)|v\rangle|^2 = 1$, $|u\rangle, |v\rangle \in \mathscr{H}$ any two unit vectors – the integral being independent on their choice, due to irreducibility of the representation [15]. With such normalization the following identities hold

$$\mathrm{Tr}_1(E) = \mathrm{Tr}_2(E) = \mathbb{1}\,, \tag{6}$$

where Tr_i denotes the partial trace over the ith Hilbert space of $\mathscr{H} \otimes \mathscr{H}$. From Eqs. (5) and (6) one has

$$A = \mathrm{Tr}_1[A \otimes \mathbb{1}\, E]\,. \tag{7}$$

Consider the *polar* parametrization $g(\psi;\boldsymbol{n}) = \exp(i\psi \boldsymbol{n} \cdot \boldsymbol{T})$ of group elements, where $\boldsymbol{T} \equiv \{T_i\}$ is a basis for the Lie algebra of $\boldsymbol{T}$, and $\boldsymbol{n} \in \boldsymbol{\Lambda}$ is a unit vector $|\boldsymbol{n}|^2 = 1$ here playing the role of λ. Using the new polar variables $\{\psi, \boldsymbol{n}\}$ for $\boldsymbol{T}$, the intertwining operator rewrites as follows

$$E = \int_{\boldsymbol{\Lambda}} \mathrm{d}\boldsymbol{n}\, E(\boldsymbol{n} \cdot \boldsymbol{T})\,, \tag{8}$$

where

$$E(\boldsymbol{n} \cdot \boldsymbol{T}) = \int \mathrm{d}\mu(\psi) \mathrm{e}^{-i\psi \boldsymbol{n} \cdot \Delta \boldsymbol{T}}\,, \tag{9}$$

$\Delta\boldsymbol{T} \equiv \boldsymbol{T} \otimes \mathbb{1} - \mathbb{1} \otimes \boldsymbol{T}$, the measure $\mathrm{d}\mu(\psi)$ includes the Jacobian $J(\psi, \boldsymbol{n})$ in $\mathrm{d}g = J(\psi, \boldsymbol{n})\mathrm{d}\psi \mathrm{d}\boldsymbol{n}$, and the integral is extended to the real axis or to a circle, for $\boldsymbol{T}$ with continuous or discrete spectrum, respectively[5]. Eqs. (7)–(9) are equivalent to the following unbiased estimation rule

$$\mathscr{E}_A(\boldsymbol{n} \cdot \boldsymbol{T}) = \mathrm{Tr}_1[A \otimes \mathbb{1}\, E(\boldsymbol{n} \cdot \boldsymbol{T})]\,. \tag{10}$$

For A traceclass, integral and trace in Eqs. (9), (10) can be exchanged, obtaining

$$\mathscr{E}_A(\boldsymbol{n} \cdot \boldsymbol{T}) = \int \mathrm{d}\mu(\psi) \mathrm{Tr}[\mathrm{e}^{-i\psi \boldsymbol{n} \cdot T} A] \mathrm{e}^{+i\psi \boldsymbol{n} \cdot \boldsymbol{T}}\,. \tag{11}$$

But, how to realize the quorum of observables in practice? In a concrete situation one doesn't have an infinite set of detectors at disposal for all possible observables in $\mathscr{Q}$. However, one can start from a finite maximal set of commuting observables, say $\{H_\nu\}$, and achieve the quorum observables by evolving H_ν under the action of a group $\boldsymbol{G}$ of physical transformations (in the Heisenberg picture). This can be attained, for example, by preceding the H_ν-detectors with an apparatus that performs the transformations of $\boldsymbol{G}$. For example, as shown in the following, for estimating angular momentum observables a quorum is given by the set of all angular momentum operators $\boldsymbol{J} \cdot \boldsymbol{n}$ on the sphere $\boldsymbol{n} \in S^2$. The detector is simply a Stern–Gerlach apparatus for J_z preceded by a uniform magnetic field in the xy plane. The magnetic field in the xy plane rotates J_z to $\boldsymbol{J} \cdot \boldsymbol{n}$. In other situations, the group $\boldsymbol{G}$ is simply achieved by tuning some parameters at detectors, e.g. rotating the phase of the local oscillator (LO) in the homodyne detector of a homodyne tomographer [11]. In this scenario the quorum manifold $\boldsymbol{\Lambda}$ is isomorphic to the coset space $\boldsymbol{G}/\boldsymbol{H}$, $\boldsymbol{H}$ denoting the stabilizer of the *seed observables* H_ν under the action of $\boldsymbol{G}$. Notice that, in the present construction, the *physical group* $\boldsymbol{G}$ is generally different from the *frame group* $\boldsymbol{T}$.

[5] More generally, when the Lie algebra of $\boldsymbol{G}$ can be decomposed into the direct sum of sub-algebras, one can exploit a separate polar parametrization for each sub-algebra, and the parameter ψ becomes a point on a cylinder/torus.

3. Examples

Before continuing, let us illustrate the above procedure on the basis of some examples. In Table 1 the estimation rule for some different apparatuses is given. The first example is homodyne tomography [11]. The measuring apparatus is a homodyne detector with tunable phase with respect to the LO. The quantum system is the harmonic oscillator representing a single mode of the e.m. field, with annihilation and creation operators $[a, a^\dagger] = 1$ acting on a infinite dimensional $\mathscr{H}$. The frame group $\boldsymbol{T}$ is the Heisenberg–Weyl group of displacement operators $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$. The quorum is the set of field quadratures $X_\phi \doteq \frac{1}{2}(a^\dagger \mathrm{e}^{i\phi} + ae^{-i\phi})$ with uniformly distributed phase $\phi \in [0, \pi]$, $\mathrm{d}\mu(\phi) = \mathrm{d}\phi/\pi$. The physical group $\boldsymbol{G}$ is the group $U(1)$ of rotations of the phase ϕ. The stabilizer $\boldsymbol{H}$ is generated by the π-rotation, which is equivalent to the quadrature inversion $X_{\phi+\pi} = -X_\phi$.

The second example in Table 1 represents the estimation of angular momentum observables for a spin-J elementary particle, or any $(2J+1)$-level sys-

Table 1
Examples of applications of the universal quantum estimation procedure. Here $\boldsymbol{T}$ denotes the *frame group*, $\boldsymbol{G}$ the *physical group*, $\boldsymbol{\Lambda}$ the *quorum* manifold, $Q_\lambda \equiv \boldsymbol{n}\cdot\boldsymbol{T}$ the *quorum* observables (both λ and $\boldsymbol{n}$ denote a point in $\boldsymbol{\Lambda}$, depending on convenience (see text). The estimation rule $E(Q_\lambda)$ is introduced in Eqs. (1) and (9)–(10). The measure $\mathrm{d}\mu(\lambda)$, is defined in Eqs. (1) and (8). *WH* denotes the Weyl Heisenberg group of displacement operators in the complex plane. D_2 denotes the dihedral group with a two-fold axis. All other notation is standard.

$\boldsymbol{T}$	WH	$WH^{\otimes(n+1)}$
$\boldsymbol{G}$	$U(1)$	$SU(n+1)$
$\boldsymbol{\Lambda}$	$[0,\pi]$	$SU(n+1)$
$Q_\lambda \equiv \boldsymbol{n}\cdot\boldsymbol{T}$	X_ϕ	$X(\boldsymbol{\theta},\boldsymbol{\psi})$
$\mathrm{d}\mu(\lambda)$	$\frac{\mathrm{d}\phi}{\pi}$	$\frac{\mathrm{d}\boldsymbol{\psi}}{(2\pi)^{n+1}}\,\mathrm{d}\boldsymbol{u}$
$E(Q_\lambda)$	$\frac{1}{2}\int_0^\infty \mathrm{d}k\,k\cos(k\Delta X_\phi)$	$\frac{1}{2}\int_0^\infty \frac{\mathrm{d}k\,k^{n+1}}{2^n n!}\cos(k\Delta X(\boldsymbol{\theta},\boldsymbol{\psi}))$
Measuring apparatus	ϕ-tunable homodyne	Mode-and-ϕ-tunable homodyne

$\boldsymbol{T}$	$SU(2)$	D_2
$\boldsymbol{G}$	$SO(3)$	D_2
$\boldsymbol{\Lambda}$	S^2	$\boldsymbol{Z}_3$
$Q_\lambda \equiv \boldsymbol{n}\cdot\boldsymbol{T}$	$\boldsymbol{J}\cdot\boldsymbol{n}$	σ_α
$\mathrm{d}\mu(\lambda)$	$\frac{\mathrm{d}\boldsymbol{n}}{4\pi}$	$\frac{1}{3}$
$E(Q_\lambda)$	$\frac{2J+1}{\pi}\int_0^\pi \mathrm{d}\psi\,\sin^2\frac{\psi}{2}\cos(\psi\Delta\boldsymbol{J}\cdot\boldsymbol{n})$	$\frac{3}{2}\sigma_\alpha\otimes\sigma_\alpha+\frac{1}{2}$
Measuring apparatus	Two-field Stern–Gerlach	Two-field Stern–Gerlach

tem. The frame group is $\boldsymbol{T}=SU(2)$, whereas the physical group is the group of rotations of the angular momentum $\boldsymbol{G}=SO(3)$. The quorum is the set of all angular momentum operators $\boldsymbol{J}\cdot\boldsymbol{n}$ on the Bloch sphere $S^2 \simeq SU(2)/U(1)$ uniformly distributed with $\mathrm{d}\mu(\boldsymbol{n})=\mathrm{d}\boldsymbol{n}/(4\pi)$. The apparatus is the two-field Stern–Gerlach machine already mentioned.

The third example is a particular case of the previous one for $J=\frac{1}{2}$. Here the discrete *minimal quorum* $\mathscr{Q}=\{\sigma_x,\sigma_y,\sigma_z\}$ of Pauli matrices is available. $\boldsymbol{T}=\boldsymbol{G}\equiv D_2$ the dihedral group of π-rotations around three perpendicular axes. Notice that this minimal quorum is not complete for estimation with $J>\frac{1}{2}$.[6]

The last example in Table 1 is the case of multimode homodyne tomography with one LO [17]. The apparatus is a homodyne detector with phase and mode-tunable LO. The quantum system is a multimode e.m. field, with annihilation operators $a_0,a_1,\dots,a_n$. The quorum is the set of quadratures $X(\boldsymbol{\theta},\boldsymbol{\psi})=\frac{1}{2}\left[A^\dagger(\boldsymbol{\theta},\boldsymbol{\psi})+A(\boldsymbol{\theta},\boldsymbol{\psi})\right]$ where $A(\boldsymbol{\theta},\boldsymbol{\psi})=\sum_{l=0}^n \mathrm{e}^{-i\psi_l}u_l(\boldsymbol{\theta})a_l$ are bosonic mode operators, with $\boldsymbol{u}\in S^{n+1}$ a point on a Poincaré hyper-sphere, $\boldsymbol{\psi}=\{\psi_l\in[0,2\pi]\}$, $\boldsymbol{\theta}=\{\theta_l\in[0,\pi/2]\}$, with probability measure $\mathrm{d}\mu(\boldsymbol{\theta},\boldsymbol{\psi})=\prod_{l=0}^n\frac{\mathrm{d}\psi_l}{2\pi}\,\mathrm{d}\boldsymbol{u}$. The annihilation operators $A(\boldsymbol{\theta},\boldsymbol{\psi})$ of the quorum are the orbit of a fixed single mode, say a_0, under the action of $\boldsymbol{G}\equiv SU(n+1)$. As a relevant example of application, here the estimator of the matrix element $\langle\{n_l\}|R|\{m_l\}\rangle$ of the full joint density operator R of modes for generally nonunit quantum efficiency η [17]

$$\mathscr{E}^{(\eta)}_{|\{m_l\}\rangle\langle\{n_l\}|}(x;\boldsymbol{\theta},\boldsymbol{\psi}) = \mathrm{e}^{-i\sum_{l=0}^{n}(n_l-m_l)\psi_l}\frac{\kappa^{n+1}}{n!} \times\prod_{l=0}^{n}\left\{\left[-i\sqrt{\kappa}\,u_l(\boldsymbol{\theta})\right]^{\mu_l-\nu_l}\sqrt{\frac{\nu_l!}{\mu_l!}}\right\} \times\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t+2i\sqrt{\kappa t}\,x}\,t^{n+\frac{1}{2}\sum_{l=0}^{n}(\mu_l-\nu_l)} \times\prod_{l=0}^{n}L_{\nu_l}^{\mu_l-\nu_l}\left[\kappa u_l^2(\boldsymbol{\theta})t\right]. \tag{12}$$

[6] Although quorum minimality maybe dictated by elegance and simplicity requirements, it is not clear if it is of any practical use. For example, achieving the complete rotation group physically is not more difficult than achieving only a discrete subgroup. Moreover, computer simulations show that statistical errors in the estimation procedure are unaffected by the size of the quorum [L. Maccone, PhD thesis]. For a reconstruction method of the spin density matrix based on a minimal quorum, see Ref. [16]

In Eq. (12) $\mu_l = \max(m_l, n_l)$, $\nu_l = \min(m_l, n_l)$, $\kappa = 2\eta/(2\eta - 1)$, and $L_n^l(x)$ denote generalized Laguerre polynomials. Other examples can be found in Ref. [17].

For *composite systems* with Hilbert space $\mathscr{H} = \otimes_{n=1}^N \mathscr{H}_n$ a quorum is the Cartesian-product quorum $\mathscr{Q} = \times_{n=1}^N \mathscr{Q}_n$, with the tensor-product estimation rule for factorized operators $\otimes_{n=1}^N A_n$: $\mathscr{E}_{\otimes_{n=1}^N A_n}(Q_1, \ldots, Q_N) = \prod_{n=1}^N \mathscr{E}_{A_n}(Q_n)$ – the rule being extended to all operators in $\mathscr{L}(\mathscr{H})$ by linearity. Notice that the tensor product rule is sufficient to estimate any *global observable* in $\mathscr{L}(\mathscr{H})$ (for example, the full joint density matrix), but a *local measurement* on each subsystem is needed, i.e. subsystems must be *distinguishable* (an example is the selfhomodyne technique in Ref. [18]). In contrast, for *indistinguishable* systems one needs a quorum of global observables. For example, for two spin-$\frac{1}{2}$ particles two perpendicular gradients allow to separate multiplet components [19]. Another example of global quorum observables is the one-LO multimode homodyne tomography mentioned above.

4. Deconvolution of instrumental noise

In the practical situation the estimation needs to be achieved in the presence of instrumental noise. In quantum mechanics instrumental noise of any kind can be described by a unit-preserving completely positive (CP) map Γ: $\mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$ (in fact, CP-maps are used to describe any quantum open system: see Ref. [20]). The noise Γ can be *deconvolved* for the estimation of A if $\mathscr{E}_A(\mathscr{Q})$ is in the domain of Γ^{-1} and $\Gamma^{-1}[\mathscr{E}_A(Q_\lambda)]$ is still a function of Q_λ (See footnote 3). In this case the ensemble average of A can be estimated in the presence of noise Γ using the *deconvolved estimator* [see Eq. (8)] $\Gamma^{-1}[\mathscr{E}_A(\boldsymbol{n}\cdot\boldsymbol{T})] = \mathrm{Tr}_1\{A \otimes \mathbb{1}\, \Gamma^{-1}[E(\boldsymbol{n}\cdot\boldsymbol{T})]\}$[7]. An example of noise deconvolution is the case of Gaussian noise in homodyne detection [21]. Here, in order to evaluate the deconvolved estimator for noise variance Δ^2, one only needs the identity $\Gamma[\exp(ikX_\phi)] = \exp(ikX_\phi - \frac{1}{4}k^2\Delta^2)$ in order to get the deconvolved estimation rule

$$\Gamma^{-1}\left[E\left(X_\phi\right)\right] = \tfrac{1}{2}\int_0^\infty \mathrm{d}k\, k\, \mathrm{e}^{1/4\Delta^2 k^2} \cos\left(k\Delta X_\phi\right). \tag{13}$$

Nonunit quantum efficiency corresponds to Gaussian noise with $\Delta^2 = (1-\eta)/(2\eta)$. Notice that, generally, there is a A-dependent bound for Δ^2, above which the deconvolution fails [21]. In the example of Eq. (12) the estimator $\mathscr{E}^{(\eta)}$ is already a deconvolved estimator, and the bound is $\eta > \frac{1}{2}$. Another example is the $J = \frac{1}{2}$ estimation in a Pauli-channel $\Gamma_p(A) = (1-p)A + \frac{p}{2}\mathrm{Tr}[A]$, $0 \le p \le 1$, with deconvolved estimation rule for $p < 1$: $\mathscr{E}_A^{(p)}(\sigma_\alpha) = \frac{3}{2}\{(1-p)^{-1}\mathrm{Tr}[A\sigma_\alpha]\sigma_\alpha + \frac{1}{2}\mathrm{Tr}[A]\}$.

In Eq. (11) the estimation rule was specialized to traceclass operators. For non-traceclass operators one can evaluate the integral in Eq. (9) as a distribution and use a kind of renormalization technique that exploits the equivalence relation $\simeq$ between estimators, dropping the unbounded null-estimator part. For example, for homodyne tomography, all null estimators are linear combinations of $X_\phi^k \mathrm{e}^{\pm i(2p+2+k)} \simeq 0$, for $k, p \ge 0$. One can deduce a function calculus based on the equivalence $\simeq$, and evaluate all the estimators of the unbounded s-ordered field monomials $\{a^{\dagger m} a^n\}_s$ for quantum efficiency η in terms of 'truncated' Hermite polynomials (for a short account of the renormalization procedure see [22]; a detailed derivation will be published elsewhere. For normal ordering the truncated-Hermite estimators are equivalent to the complete Hermite given in [23]). For example, the estimator of the photon number is simply $\mathscr{E}_{a^\dagger a}^{(\eta)}(X_\phi) = 2X_\phi^2 - 1/(2\eta)$, $\mathscr{E}^{(\eta)}$ denoting deconvolved estimators for quantum efficiency η. The same technique can be used for the case of one-LO multimode homodyne tomography. For example, for the total photon number of two modes one has $\mathscr{E}_{a^\dagger a + b^\dagger b}^{(\eta)}(X(\boldsymbol{\theta}, \boldsymbol{\psi})) = 4X(\boldsymbol{\theta}, \boldsymbol{\psi})^2 - 1/\eta$.

5. Estimation strategies

The group theoretical quorum here derived can also be used with other estimation strategies different

[7] Noise can be deconvolved more generally when there is a new quorum $\mathscr{Q}_\Gamma$ isomorphic to $\mathscr{Q}$ with a map m_Γ: $\mathscr{Q} \leftrightarrow \mathscr{Q}_\Gamma$ such that $\Gamma^{-1}[\mathscr{E}_A(Q_\lambda)]$ shares the same spectral decomposition of $m_\Gamma(Q_\lambda)$ for all $Q_\lambda \in \mathscr{Q}$.

from the present averaging procedure. Thanks to the existence of null estimators, there are many equivalent unbiased estimators, and *adaptive* least-squares methods are possible [24], with the estimator 'adapted' to the set of measured data, by minimizing the r.m.s. error in the equivalence class. Also, *active adaptive* approaches can be exploited as well, where the measure probability $d\mu(\lambda)$ over the quorum is upgraded from the uniform one, while accumulating data. Another relevant strategy, the max-likelihood method, can be used for measuring unknown parameters of a unitary transformation on a given state, or for measuring the matrix elements of the density operator itself [25]. For the full joint density matrix R the likelihood function is $\mathscr{L} = \sum_i \log\{\sum_{kn} |\langle n|TW_k^\dagger|q_i\rangle_{\lambda_i}|^2\} - \beta\, \mathrm{Tr}(T^\dagger T)$, where β is a Lagrange multiplier, $|n\rangle$ any basis for $\mathscr{H}$, the sum runs over the label of the ith measurement, T is an upper triangular matrix in the Cholesky decomposition $R = T^\dagger T$ of the density matrix R, W_k, with $\sum_k W_k^\dagger W_k = \mathbb{1}$, give the Kraus decomposition $\Gamma[A] = \sum_k W_k^\dagger A W_k$ of the noise Γ, and finally $|q\rangle_\lambda$ denotes an eigenvector of the quorum observable Q_λ with eigenvalue q. Notice that, since this method needs a finite parametrization of the density matrix, truncation of the Hilbert space dimension is unavoidable, and the estimation becomes biased. However, smaller statistical errors are obtained, as compared to the averaging procedure of this letter [25].

6. Further developments

Square integrable representations R have been considered for simplicity up to now. For non square integrable representations the method of imprimitivity systems can be used [26]. This allows, for example, to consider the case of the full Poincarè group, corresponding to the general quantum estimation for a relativistic particle (the measuring apparatus becoming a Mőssbauer variant of the Stern–Gerlach apparatus previously considered, where, in addition to the angular momentum, the energy of the particle is measured in a moving frame). Imprimitivity systems allow also include estimation methods of the kind of the photon number tomography of Ref. [27]. Generalization to non unimodular groups is also possible. Here the partial traces in Eq. (6) give no longer the identity, whereas Eq. (5) holds for either $A = \mathbb{1}$ or $B = \mathbb{1}$, depending if dg is right or left invariant, respectively. When the operators $\mathrm{Tr}_n(E)$ can be inverted, Eq. (7) – the core of the present method – can be easily generalized.

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References

[1] B. d'Espagnat, Conceptual Foundations of Quantum Mechanics, Addison-Wesley, Reading, Massachusetts, 1989.
[2] C.W. Helstrom, Quantum Detection and Estimation Theory, Academic Press, New York, 1976.
[3] E. Arthurs, J.L. Kelly, Bell. Syst. Tech. J. 44 (1965) 725.
[4] J. Mod. Opt. 44 (1997) (special issue).
[5] D.-G. Welsch, W. Vogel, T. Opatrnỳ, Progr. Opt. (1999), review in press.
[6] U. Fano, Rev. Mod. Phys. 29 (1957) 74.
[7] D.T. Smithey, M. Beck, M.G. Raymer, A. Faridani, Phys. Rev. Lett. 70 (1993) 1244.
[8] C. Kurtsiefer, T. Pfau, J. Mlynek, Nature 386 (1997) 150.
[9] K. Vogel, H. Risken, Phys. Rev. A 40 (1989) 2847.
[10] G.M. D'Ariano, C. Macchiavello, M.G.A. Paris, Phys. Rev. A 50 (1994) 4298.
[11] G.M. D'Ariano, Measuring Quantum States, in: T. Hakioglu, A.S. Shumovsky (Eds.), Quantum Optics and Spectroscopy of Solids, Kluwer Academic Publishers, Dordrecht, 1997, p. 175.
[12] G. Breitenbach, S. Schiller, J. Mlynek, Nature 387 (1997) 471.
[13] G.M. D'Ariano, in: M. Caves, O. Hirota, A.S. Holevo (Eds.), Quantum Communication, Computing, and Measurement, Plenum Publishing, New York and London, 1997, p. 253.
[14] G.M. D'Ariano, C. Macchiavello, Phys. Rev. A 57 (1998) 3131.
[15] H.F. Jones, Groups, Representations and Physics, IOP Publishing, Bristol Philadelphia, 1998.
[16] J.-P. Amiet, S. Weigert, J. Phys. A: Math. Gen. 32 (1999) L269.
[17] G.M. D'Ariano, M.F. Sacchi, P. Kumar, Phys. Rev. A 61 (2000) 13806.
[18] G.M. D'Ariano, M. Vasilyev, P. Kumar, Phys. Rev. A 58 (1998) 636.

[19] M. Paini, unpublished.
[20] E.B. Davies, Quantum Theory of Open Systems, Academic Press, New York, 1973.
[21] G.M. D'Ariano, N. Sterpi, J. Mod. Opt. 44 (1997) 2227.
[22] G.M. D'Ariano, in: P. Kumar, G. D'Ariano, O. Hirota (Eds.), Quantum Communication, Computing, and Measurement, Plenum Publishing, New York and London, 2000, p. 137.
[23] T. Richter, Phys. Rev. A 53 (1996) 1197.
[24] G.M. D'Ariano, M.G.A. Paris, Phys. Rev. A 60 (1999) 518.
[25] K. Banaszek, G.M. D'Ariano, M.G.A. Paris, M.F. Sacchi, Phys. Rev. A 61 (2000) 010304.
[26] G. Cassinelli, G.M. D'Ariano, E. De Vito, A. Levrero, unpublished.
[27] K. Banaszek, K. Wodkiewicz, Phys. Rev. Lett. 76 (1996) 4344.