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Quantum Tomography: General Theory and New Experiments

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Abstract

Quantum tomography is a general method for estimating the ensemble average of all operators of an arbitrary quantum system from a set of measurements of a *quorum* of observables. A procedure for deconvolving instrumental noise is available, which makes the tomographic method feasible in different physical contexts. Recent developments are presented and new experiments are proposed, which are now made possible by the tomographic technique.

1. Introduction

Suppose that we can prepare a quantum system repeatedly in the same state, and make a series of experiments such that we can measure a different observable in each experimental setting. Can we estimate the ensemble average of any desired system operator – including the density matrix of the state itself – from the set of measured outcomes? How the measured observables must be chosen in order to allow such universal estimation? The answer to this question is given by the method of *quantum tomography.* Similarly to classical tomography, where a picture of a hidden object is build up using various observations from different angles, in quantum tomography a complete description of a quantum system is recovered by observing complementary features in different experimental runs. For example, in quantum optics, optical homodyne tomography allows to reconstruct the quantum state of light from a set of measured field quadratures at different phases with respect to the homodyne local oscillator (LO). This is analogous to what happens in computer-assisted tomography, where the image of a hidden internal part of a living body is obtained from recorded transmission profiles of X-rays that penetrated the body from various directions. The name *quantum tomography* originated in quantum optics from such analogy with the classical computer-assisted tomography. There, the set of quadrature probability distributions for varying LO-phase was recognized [1] to be the Radon transform of the Wigner function, the Radon transform being the basic imaging tool in medical tomography. This method led to a first qualitative technique for measuring the matrix elements of the radiation density operator [2]. A first quantitative technique has then been presented [3, 4], the one which is now used in the lab [5]. The method has then been generalized to the estimation of arbitrary field observable [6], and, recently, it has been extended to arbitrary quantum system [7, 8].

In this paper, after reviewing the general tomographic approach in Section 2, in Sect. 3 a set of new experiments will be given, which are now made possible by the general quantum tomographic technique. Particular attention is devoted to a new class of experiments, which allow, for the first time, to check experimentally the quantum mechanical state-reduction rule.

2. The general approach

The set of quadratures in homodyne tomography is an example of *quorum* [9] of observables, namely a "complete" set of noncommuting observables for determining the quantum state of the system. The concept of quorum is the basis of the general tomographic method for estimating the ensemble average of all operators of arbitrary quantum system. As we will see in the following, the method provides a concrete framework to design measuring apparatuses for estimation, taking into account also instrumental noise of any kind in the measurement. In this section the method will be briefly reviewed at an intermediate level of generality: routes to generalizations can be found in Ref. [7].

2.1 Quorum and estimating procedure

We call the set $\mathcal{Q} = \{Q_\lambda\}$ of observables Q_λ, $\lambda \in \Lambda$, a *quorum* for the quantum system $\mathcal{S}$ if it is possible to estimate the ensemble average $\langle A \rangle$ of any operator $A \in \mathcal{L}(\mathcal{H})$ by using only measurement outcomes of quorum observables. An *unbiased estimation rule* $\mathcal{E}$ for the quorum $\mathcal{Q}$ assigns to every operator A an operator valued function of $Q_\lambda \in \mathcal{Q}$, the *unbiased estimator* $\mathcal{E}_A(Q_\lambda)$, such that the ensemble average $\langle A \rangle = \mathrm{Tr}\,[A\varrho]$ for arbitrary *unknown* state ϱ can be obtained by averaging over the quorum as follows

$$\langle A \rangle = \int_{\boldsymbol{\Lambda}} d\mu(\lambda)\, \langle \mathcal{E}_A(Q_\lambda) \rangle\,, \tag{2.1}$$

where μ is a probability measure over $\boldsymbol{\Lambda}$, and the integral is a sum for discrete set $\boldsymbol{\Lambda}$ (explicit dependence of $\mathcal{E}_A$ on λ is omitted for simplicity of notation).

Eq. (2.1) corresponds to the following *estimation procedure* for $\langle A \rangle$: i) select an observable Q_λ randomly in the quorum $\mathcal{Q}$ according to the probability measure μ; ii) measure Q_λ and evaluate the function $\mathcal{E}_A$ of the outcome; iii) average the result over many measurements with different $Q_\lambda \in \mathcal{Q}$. Notice that the ensemble average $\langle A \rangle$ of any operator $A \in \mathcal{L}(\mathcal{H})$ is obtained from the *same set of data* using the same fixed estimation rule.

Eq. (2.1) must be true for arbitrary ϱ, whence

$$A = \int_{\boldsymbol{\Lambda}} d\mu(\lambda)\, \mathcal{E}_A(Q_\lambda)\,, \tag{2.2}$$

with integral convergence in expectation. Notice that the estimation rule is generally not unique, since there exist *null estimators* $\mathcal{N}$ over $\mathcal{Q}$ satisfying the identity

$$\int_{\boldsymbol{\Lambda}} d\mu(\lambda)\, \mathcal{N}(Q_\lambda) = 0\,. \tag{2.3}$$

The existence of null estimators sets an equivalence relation $\simeq$ between unbiased estimators (two estimators are equivalent if they differ by a null estimator).

2.2 The general unbiased estimation rule

I now derive a general estimation rule abstractly: physical implementations and apparatuses will be considered in the next subsections. An unbiased estimation rule is obtained from any (Lie) group $\boldsymbol{T}$ of transformations $g \in \boldsymbol{T}$ with invariant measure dg and unitary irreducible representation (UIR) R over the Hilbert space $\mathcal{H}$ of the quantum system ($\boldsymbol{T}$ unimodular for simplicity). The following selfadjoint involution $E = E^\dagger \equiv E^{-1}$ on $\mathcal{H} \otimes \mathcal{H}$

$$E = \int_{\boldsymbol{T}} dg\, R^\dagger(g) \otimes R(g)\,, \tag{2.4}$$

is an *intertwining operator*, namely, for any two operators A and B one has the identity

$$E(A \otimes B) = (B \otimes A)\, E\,, \tag{2.5}$$

which is easily proved by the first Schur lemma. For square-integrable R, the invariant measure dg can be normalized as $\int_{\boldsymbol{T}} dg\, |\langle u|\, R(g)\, |v\rangle|^2 = 1$, $|u\rangle, |v\rangle \in \mathcal{H}$ any two unit vectors – the integral being independent on their choice, due to irreducibility of the representation. With such normalization the following identities hold

$$\mathrm{Tr}_1\,(E) = \mathrm{Tr}_2\,(E) = \hat{1}\,, \tag{2.6}$$

where Tr_i denotes the partial trace over the ith Hilbert space of $\mathcal{H} \otimes \mathcal{H}$. From Eqs. (2.5) and (2.6) one has

$$A = \mathrm{Tr}_1\,[A \otimes \hat{1} E]\,. \tag{2.7}$$

Consider the *polar* parametrization $g(\psi; \vec{n}) = \exp\,(i\psi\vec{n} \cdot \vec{T})$ of group elements, where $\vec{T} \equiv \{T_i\}$ is a basis for the Lie algebra of $\boldsymbol{T}$, and $\vec{n} \in \boldsymbol{\Lambda}$ is a unit vector $|\vec{n}|^2 = 1$ here playing the role of λ. Using the new polar variables $\{\psi, \vec{n}\}$ for $\boldsymbol{T}$, the intertwining operator rewrites as follows

$$E = \int_{\Lambda} d\vec{n}\, E(\vec{n} \cdot \vec{T})\,, \tag{2.8}$$

where

$$E(\vec{n} \cdot \vec{T}) = \int d\mu(\psi)\, e^{-i\psi\vec{n}\cdot\Delta\vec{T}}\,, \tag{2.9}$$

$\Delta\vec{T} \equiv \vec{T} \otimes \hat{1} - \hat{1} \otimes \vec{T}$, the measure $d\mu(\psi)$ includes the Jacobian $J(\psi, \vec{n})$ in $dg = J(\psi, \vec{n})\, d\psi\, d\vec{n}$, and the integral is extended to the real axis or to a circle, for $\vec{T}$ with continuous or discrete spectrum, respectively. Eqs. (2.7–2.9) are equivalent to the following unbiased estimation rule

$$\mathcal{E}_A(\vec{n} \cdot \vec{T}) = \mathrm{Tr}_1\,[A \otimes \hat{1}\, E(\vec{n} \cdot \vec{T})]\,. \tag{2.10}$$

For A traceclass, integral and trace in Eqs. (2.9) (2.10) can be exchanged, obtaining

$$\mathcal{E}_A(\vec{n} \cdot \vec{T}) = \int d\mu(\psi)\, \mathrm{Tr}\,[e^{-i\psi\vec{n}\cdot T} A]\, e^{+i\psi\vec{n}\cdot\vec{T}}\,. \tag{2.11}$$

2.3 Physical implementation

How to realize the quorum of observables in practice? In a concrete situation one doesn't have an infinite set of detectors at disposal for all possible observables in $\mathcal{Q}$. However, one can start from a finite maximal set of commuting observables, say $\{H_\nu\}$, and achieve the quorum observables by evolving H_ν under the action of a group $\boldsymbol{G}$ of physical transformations (in the Heisenberg picture). This can be attained, for example, by preceding the H_ν-detectors with an apparatus that performs the transformations of $\boldsymbol{G}$. For example, as shown in the following, for spin tomography a quorum is given by the set of all angular momentum operators $\vec{J} \cdot \vec{n}$ on the sphere $\vec{n} \in S^2$. The detector is a Stern-Gerlach apparatus for J_z preceded by a uniform magnetic field in the *xy* plane. The magnetic field in the *xy* plane rotates J_z to $\vec{J} \cdot \vec{n}$. In other situations, the group $\boldsymbol{G}$ is simply achieved by tuning some parameters at detectors, e.g. rotating the phase of the local oscillator (LO) in the homodyne detector of a homodyne tomographer [4]. In this scenario the quorum manifold Λ is isomorphic to the coset space $\boldsymbol{G}/\boldsymbol{H}$, $\boldsymbol{H}$ denoting the stabilizer of the *seed observables* H_ν under the action of $\boldsymbol{G}$. Notice that, in the present construction, the *physical group* $\boldsymbol{G}$ is generally different from the *tomographic group* $\boldsymbol{T}$.

2.4 Examples

In this subsection the general tomographic procedure is illustrated on the basis of some examples. In Table 1 the estimation rule for some different apparatuses is given. The first example is homodyne tomography [4]. The measuring apparatus is a homodyne detector

Table 1

Examples of applications of the general quantum tomographic method. Here $\boldsymbol{T}$ denotes the *tomographic group*, $\boldsymbol{G}$ the *physical group*, $\boldsymbol{\Lambda}$ the *quorum* manifold, $Q_\lambda \equiv \vec{n} \cdot \vec{T}$ the *quorum* observables (on convenience, both λ and $\vec{n}$ denote a point in Λ). The estimation rule $E(Q_\lambda)$ is given in Eqs. (2.1) and (2.9−2.10). The measure $d\mu(\lambda)$, is defined in Eqs. (2.1) and (2.8). WH denotes the Weyl Heisenberg group of displacement operators in the complex plane. D_2 denotes the dihedral group with a two-fold axis. $P_n\boldsymbol{C} \simeq SU(n+1)/U(n)$ denotes the n-dimensional complex projective space. All other notation is standard.

	Homodyme tomography	Multimode homodyme
$\boldsymbol{T}$	WH	$SU(n+1)$
$\boldsymbol{G}$	$U(1)$	$SU(n+1)$
$\boldsymbol{\Lambda}$	$[0,\pi]$	$P_n\boldsymbol{C}$
$Q_\lambda \equiv \vec{n} \cdot \vec{T}$	X_ϕ	$X(\theta,\psi)$
$d\mu(\lambda)$	$\dfrac{d\phi}{\pi}$	$\dfrac{d\vec{\psi}}{(2\pi)^{n+1}}\, d\vec{u}$
$E(Q_\lambda)$	$\dfrac{1}{2}\displaystyle\int_0^\infty dk\, k \cos(k\, \Delta X_\phi)$	$\dfrac{1}{2}\displaystyle\int_0^\infty \dfrac{dk\, k^{n+1}}{2^n n!} \cos(k\, \Delta X(\theta,\psi))$

	Spin tomography	Pauli tomography
$\boldsymbol{T}$	$SU(2)$	D_2
$\boldsymbol{G}$	$SO(3)$	D_2
$\boldsymbol{\Lambda}$	S^2	$\boldsymbol{Z}_3$
$Q_\lambda \equiv \vec{n} \cdot \vec{T}$	$\vec{J} \cdot \vec{n}$	σ_α
$d\mu(\lambda)$	$\dfrac{d\vec{n}}{4\pi}$	$\dfrac{1}{3}$
$E(Q_\lambda)$	$\dfrac{2J+1}{n}\displaystyle\int_0^\pi d\psi \sin^2 \dfrac{\psi}{2} \cos(\psi\, \Delta\vec{J} \cdot \vec{n})$	$\dfrac{3}{2}\, \sigma_\alpha \otimes \sigma_\alpha + \dfrac{1}{2}$

with tunable phase with respect to the LO. The quantum system is the harmonic oscillator representing a single mode of the e.m. field, with annihilation and creation operators $[a, a^\dagger] = 1$ acting on a infinite dimensional $\mathcal{H}$. The tomographic group $\boldsymbol{T}$ is the Heisenberg-Weyl group of displacement operators $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$. The quorum is the set of field quadratures $X_\phi \doteq \frac{1}{2}(a^\dagger e^{i\phi} + a\, e^{-i\phi})$ with uniformly distributed phase $\phi \in [0, \pi]$, $d\mu(\phi) = d\phi/\pi$. The physical group $\boldsymbol{G}$ is the group $U(1)$ of rotations of the phase ϕ. The stabilizer $\boldsymbol{H}$ is generated by the π-rotation, which is equivalent to the quadrature inversion $X_{\phi+\pi} = -X_\phi$.

The second example in Table 1 is the case of multimode homodyne tomography with one LO[10]. The apparatus is a homodyne detector with phase and mode-tunable LO. The quantum system is a multimode e.m. field, with annihilation operators $a_0, a_1, \ldots, a_n$. The quorum is the set of quadratures $X(\theta, \psi) = \frac{1}{2}\,[A^\dagger(\theta, \psi) + A(\theta, \psi)]$ where $A(\theta, \psi) = \sum_{l=0}^n e^{-i\psi_l}\, u_l(\theta)\, a_l$ are bosonic mode operators, with $\vec{u} \in S^{n+1}$ a point on a Poincaré hyper-sphere, $\psi = \{\psi_l \in [0, 2\pi]\}$, $\theta = \{\theta_l \in [0, \pi/2]\}$, with probability measure $d\mu(\theta, \psi) = \prod_{l=0}^n \frac{d\psi_l}{2\pi}\, d\vec{u}$. The annihilation operators $A(\theta, \psi)$ of the quorum are the orbit of a fixed single mode, say a_0, under the action of $\boldsymbol{G} = \boldsymbol{T} \equiv SU(n+1)$. As a relevant example of application, here I report the estimator of the matrix element $\langle \{n_l\}|\, R\, |\{m_l\}\rangle$ of the full joint density operator R of modes for generally nonunit quantum efficiency η [10]

$$\mathcal{E}^{(\eta)}_{|\{m_l\}\rangle\langle\{n_l\}|}(x; \theta, \psi) = e^{-i\sum_{l=0}^{n}(n_l - m_l)\psi_l} \times \frac{\kappa^{n+1}}{n!} \prod_{l=0}^{n} \left\{ [-i\sqrt{\kappa} u_l(\theta)]^{\mu_l - \nu_l} \sqrt{\frac{\nu_l!}{\mu_l!}} \right\}$$
$$\times \int_0^\infty dt\, e^{-t + 2i\sqrt{\kappa t}\, x}\, t^{n + \frac{1}{2}\sum_{l=0}^{n}(\mu_l - \nu_l)} \prod_{l=0}^{n} L_{\nu_l}^{\mu_l - \nu_l}[\kappa u_l^2(\theta)\, t]\,. \qquad (2.12)$$

In Eq. (2.12) $\mu_l = \max(m_l, n_l)$, $\nu_l = \min(m_l, n_l)$, $\kappa = 2\eta/(2\eta - 1)$, and $L_n^l(x)$ denote generalized Laguerre polynomials. Other examples can be found in Ref. [10].

The third example in Table 1 represents the case of spin tomography for a spin-J elementary particle, or any $(2J+1)$-level system. The tomographic group is $\boldsymbol{T} = SU(2)$, whereas the physical group is the group of rotations of the angular momentum $\boldsymbol{G} = SO(3)$. The quorum is the set of all angular momentum operators $\vec{J} \cdot \vec{n}$ on the Bloch sphere $S^2 \simeq SU(2)/U(1)$ uniformly distributed with $d\mu(\vec{n}) = \mathrm{d}\vec{n}/(4\pi)$. The apparatus is the two-field Stern-Gerlach machine already mentioned.

The last example is a particular case of the previous one for $J = \frac{1}{2}$. Here the discrete *minimal quorum* $\mathcal{Q} = \{\sigma_x, \sigma_y, \sigma_z\}$ of Pauli matrices is available. $\boldsymbol{T} = \boldsymbol{G} \equiv D_2$ the dihedral group of π-rotations around three perpendicular axes. Notice that this minimal quorum is not complete for estimation with $J > \frac{1}{2}$. However, although quorum minimality maybe dictated by elegance and simplicity requirements, it is not clear if it is of any practical use. For example, achieving the complete rotation group physically is not more difficult than achieving only a discrete subgroup. Moreover, computer simulations show that statistical errors in the estimation procedure are unaffected by the size of the quorum [11].

2.5 Composite systems: distinguishable and indistinguishable subsystems

For *composite systems* with Hilbert space $\mathcal{H} = \otimes_{n=1}^N \mathcal{H}_n$ a quorum is the Cartesian-product quorum $\mathcal{Q} = \times_{n=1}^N \mathcal{Q}_n$, with the tensor-product estimation rule for factorized operators $\otimes_{n=1}^N A_n : \mathcal{E}_{\otimes_{n=1}^N A_n}(Q_1, \ldots, Q_N) = \prod_{n=1}^N \mathcal{E}_{A_n}(Q_n)$ – the rule being extended to all operators in $\mathcal{L}(\mathcal{H})$ by linearity. Notice that the tensor product rule is sufficient to estimate any *global observable* in $\mathcal{L}(\mathcal{H})$ (for example, the full joint density matrix), but a *local measurement* on

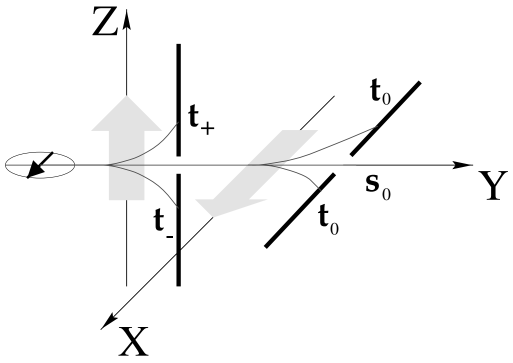

Fig. 1: Double gradient Stern-Gerlach setup for measuring the state of two indistinguishable spin-1/2 particles. From traces $t_\pm$ and t_0 one tomographically reconstructs the triplet matrix block, whereas from trace s_0 one gets the singlet matrix element. The first uniform magnetic field in the xy plane is used to rotate the angular momentum to achieve the quorum.

each subsystem is needed, i.e. subsystems must be *distinguishable* (an example is the self-homodyne technique in Ref. [12]). In contrast, for *indistinguishable* systems one needs a quorum of global observables. An example of global quorum observables is the one-LO multimode homodyne tomography already mentioned. As another example, one can consider a cluster of two identical spin-1/2 particles. For indistinguishable particles one can prove [11] that the spin reduced density matrix is block diagonal in the total spin S, e.g. there are no off-diagonal elements between singlet and triplet for two spin-$1/2$ particles. Therefore, it is possible to reconstruct the complete reduced spin density matrix through spin tomography on the total spin. In Fig. 1 a double gradient Stern-Gerlach setup is represented which allows measuring the state of two indistinguishable spin-$1/2$ particles. From traces $t_\pm$ and t_0 one can tomographically reconstruct the triplet matrix block, whereas from trace s_0 one gets the singlet matrix element [11]. For larger numbers of particles an increasing number of magnetic gradients would be needed in the tomographic reconstruction.

2.6 Taking into account instrumental noise

In the practical situation the estimation needs to be achieved in the presence of instrumental noise. In quantum mechanics instrumental noise of any kind can be described by a unit-preserving completely positive (CP) map Γ: $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$. The noise Γ can be *deconvolved* for the estimation of A if $\mathcal{E}_A(\mathcal{Q})$ is in the domain of Γ^{-1} and $\Gamma^{-1}[\mathcal{E}_A(Q_\lambda)]$ is still a function of Q_λ ($\mathcal{E}_A(Q_\lambda)$ is a function of Q_λ in the sense that it shares the same spectral decomposition of Q_λ). In this case the ensemble average of A can be estimated in the presence of noise Γ using the *deconvolved estimator* [see Eq. (2.8)] $\Gamma^{-1}[\mathcal{E}_A(\vec{n}\cdot\vec{T})] = \mathrm{Tr}_1\,\{A \otimes \hat{1}\Gamma^{-1}[E(\vec{n}\cdot\vec{T})]\}$. Noise can be deconvolved more generally when there is a new quorum $\mathcal{Q}_\Gamma$ isomorphic to $\mathcal{Q}$ with a map m_Γ: $\mathcal{Q} \leftrightarrow \mathcal{Q}_\Gamma$ such that $\Gamma^{-1}[\mathcal{E}_A(Q_\lambda)]$ shares the same spectral decomposition of $m_\Gamma(Q_\lambda)$ for all $Q_\lambda \in \mathcal{Q}$.

An example of noise deconvolution is the case of Gaussian noise in homodyne detection [13]. Here, in order to evaluate the deconvolved estimator for noise variance Δ^2, one only needs the identity $\Gamma[\exp(ikX_\phi)] = \exp(ikX_\phi - \frac{1}{4}k^2\Delta^2)$ in order to get the deconvolved estimation rule

$$\Gamma^{-1}[E(X_\phi)] = \frac{1}{2}\int_0^\infty dk\, k\, e^{\frac{1}{4}\Delta^2 k^2}\cos(k\,\Delta X_\phi)\,. \tag{2.13}$$

Nonunit quantum efficiency corresponds to Gaussian noise with $\Delta^2 = (1-\eta)/(2\eta)$. Notice that, generally, there is a A-dependent bound for Δ^2, above which the deconvolution fails

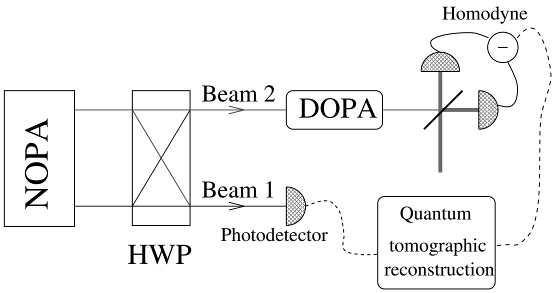

Fig. 2: Example of conditioned tomography: *reconstruction of Schrödinger-cat states of radiation* [18] (tomographic improvement of a scheme proposed in Ref. [19]). In the general conditioned tomographic scheme one has a couple of correlated beams, and a quantum measurement is performed on beam 1, while a tomographic reconstruction is made on beam 2, conditioned on the result of the measurement on beam 1. In the scheme for reconstructing Schrdinger-cat states two orthogonally polarized modes of radiation – the "signal" and "readout"–are entangled by a parametric amplifier followed by a half-wave plate. The parametric amplifier generates a correlated state of the two modes and the half wave plate rotates the polarization directions. After detection of the readout beam 1, the signal beam 2 is reduced to a Schrödinger-cat state (the degenerate parametric amplifier (DOPA) "stretches" the cat-state, making the two coherent components more distinguishable). The cat-state on beam 2 is then tomographically reconstructed through conditional homodyne tomography.

[13]. In the example of Eq. (2.12) the estimator $\mathcal{E}^{(\eta)}$ is already a deconvolved estimator, and the bound is $\eta > \frac{1}{2}$. Another example is the $J = \frac{1}{2}$ estimation in a Pauli-channel $\Gamma_p(A) = (1-p)A + \frac{p}{2}\,\mathrm{Tr}\,[A]$, $0 \le p \le 1$, with deconvolved estimation rule for $p < 1$

$$\mathcal{E}_A^{(p)}(\sigma_\alpha) = \frac{3}{2}\,\{(1-p)^{-1}\,\mathrm{Tr}\,[A\sigma_\alpha]\,\sigma_\alpha + \frac{1}{2}\mathrm{Tr}\,[A]\}\ . \tag{2.14}$$

2.7 Estimation for unbounded operators

In Eq. (2.11) the estimation rule was specialized to traceclass operators. For nontraceclass operators one can evaluate the integral in Eq. (2.9) as a distribution and use a kind of renormalization technique that exploits the equivalence relation $\simeq$ between estimators, dropping the unbounded null-estimator part. For example, for homodyne tomography, all null estimators are linear combinations of $X_\phi^k\, e^{\pm i(2p+2+k)} \simeq 0$, for $k, p \ge 0$. One can deduce a function calculus based on the equivalence $\simeq$, and evaluate all the estimators of the unbounded s-ordered field monomials $\{a^{\dagger m}a^n\}_s$ for quantum efficiency η in terms of "truncated" Hermite polynomials [14]. For example, the estimator of the photon number is simply $\mathcal{E}^{(\eta)}_{a^\dagger a}(X_\phi) = 2X_\phi^2 - 1/(2\eta)$, $\mathcal{E}^{(\eta)}$ denoting deconvolved estimators for quantum efficiency η. The same technique can be used for the case of one-LO multimode homodyne tomography. For example, for the total photon number of two modes one has $\mathcal{E}^{(\eta)}_{a^\dagger a + b^\dagger b}(X(\theta,\psi)) = 4X(\theta,\psi)^2 - 1/\eta$.

3 New experiments

The quantum tomographic technique opens new perspectives for testing quantum mechanics. For example, it is possible to directly test the nonclassicality on various one-mode

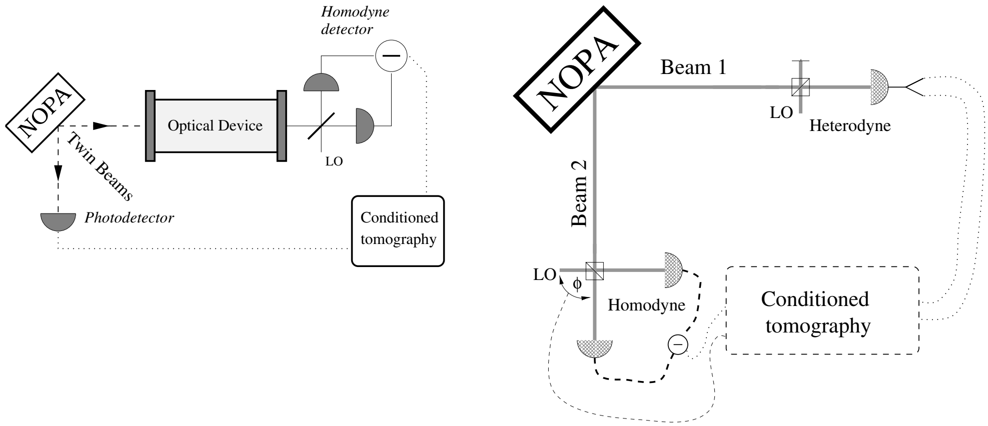

Fig. 3: Examples of conditioned tomography. On the left: *tomographic scheme for measuring the quantum Liouvillian matrix of an optical phase-insensitive device* [20]. On the right: *testing the quantum state-reduction rule* [21]. In the tomographic scheme for measuring the quantum Liouvillian matrix a random-n Fock state $|n\rangle$ for the input beam of the optical device is achieved by performing photodetection on the other twin beam, n being the measured number of photons. A non degenerate optical parametric amplifier (NOPA) with vacuum input is used to produce the twin beams. By scanning the set of states at the device input and comparing them with their respective output states, it is possible to reconstruct the Liouvillian of the device. In the scheme for testing the quantum state-reduction rule (on the right), a different kind of measurement−photodetection, heterodyne (as in figure), etc. − can be performed on beam 1, while the reduced state of beam 2 is tomographically reconstructed, conditioned by the measurement outcome on beam 1. For example, for heterodyne detection, after heterodyning beam 1, the reduced state of beam 2 is tomographically reconstructed conditioned by the heterodyne outcome. In place of the heterodyne detector one can put any other kind of detector for testing the state-reduction on different observables: for example, for heterodyne detection the reduced state is a coherent state, whereas, for photodetection, the reduced state is a number eigenstate. The state-reduction can be tested by a direct measurement of the fidelity between the theoretically expected reduced state and the experimental one, using a suitable conditioned estimator that takes into account also state distortion due to finite gain at the NOPA and nonunit quantum efficiencies at detectors. Monte Carlo simulated experiments [21] show that a decisive test can be performed even with only a few thousand measurements, with low gains at the NOPA and low quantum efficiencies at the readout photodetector.

and two-modes states, by tomographically measuring some special observables of the field [15]. Moreover, new tests of Bell's inequalities are now possible [16], based on two-mode homodyne tomography, with the possibility of achieving very good detection quantum efficiencies. Finally, using three polarization-tunable homodyne detectors, in principle it is possible to make a complete tomographic test of the preparation of a Greenberger-Horne-Zeilinger state [17], which cannot be checked by simple coincidence measurements.

3.1 Conditioned homodyne tomography

Using parametric downconversion, a new set of experiments based on *conditioned homodyne tomography* is now possible. The general conditioned scheme is as follows. A nondegenerate optical parametric amplifier (NOPA) produces a couple of correlated twin beams 1 and 2 from vacuum downconversion. A quantum measurement is performed on beam 1, and a homodyne tomographic reconstruction is made on beam 2, conditioned on the result ν of the first measurement, namely using an estimator $\mathcal{E}_A^{(\eta)}(X_\phi;\nu)$ which depends on the outcome ν of the measurement on beam 1. Examples of this conditioned tomographic scheme are: i) the tomographic reconstruction of Schrdinger-cat states of radiation of Ref. [18], which represents a tomographic improvement of a scheme proposed in Ref. [19] (see Fig. 2); ii) the tomographic scheme of Ref. [20] for measuring the quantum Liouvillian matrix of an optical phase-insensitive device (see Fig. 3); iii) the test of the quantum state-reduction rule of Ref. [21] (see Fig. 3). Especially interesting is the test of the state-reduction rule in Fig. 3, since, in our knowledge, it has never been performed. Notice that such a test is generally not equivalent to a test of the repeatability hypothesis, since the latter holds only for measurements of observables described by self-adjoint operators with discrete spectrum, whereas, for heterodyne detection, the measurement is not repeatable, as the reduced states are coherent states, which are not orthogonal.

3.2 Test of the minimum disturbance principle

Another interesting test of quantum mechanics that can be performed using quantum tomography is the Lüders state-reduction rule for degenerate observables. The rule states that for measurement outcome x the reduced state is given by

$$\varrho \to \varrho_x = \frac{E(x)\,\varrho E(x)}{\mathrm{Tr}\,[\varrho E(x)]}\,, \tag{3.15}$$

where $E(x) = \sum_d E_d(x)$ represents the projector on the degenerate eigenspace corresponding to eigenvalue x, d denoting a degeneration index. Notice that if one considers the measurement as complete, i.e. with d as unread but knowable in principle, the "marginal" rule would be obtained: $\varrho \to \varrho_x = \sum_d E_d(x)\,\varrho E_d(x)/\mathrm{Tr}\,[\sum_d \varrho E_d(x)]$, which gives an incoherent superposition as opposite to the coherent one in Eq. (3.15). This means that the Luders rule corresponds to a kind of minimum disturbance principle. A test of the Luders rule would need a way to distinguish between a pure state and a mixed one, and this can be done by quantum tomography.

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