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To take a (binary) decision you'd better use entanglement

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Abstract
We address two-mode quantum interferometry as binary measurements aimed at determining whether or not a phase perturbation has occurred. We show that optimized measurements achieve the best sensitivity when the input state is *entangled*. A concrete set-up based on parametric sources of entanglement and photodetection is also suggested and shown to approach ideal sensitivity.

Keywords: Quantum interferometry, entanglement, hypothesis testing

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Interferometry is a measurement scheme devised to monitor a given configuration and to detect minute perturbations. Usual interferometric set-ups involve two modes of radiation, arranged in such a way that any perturbation results in a two-mode phase-shift transformation of the given input. The phase shift is monitored by probing the output, and the set-up is optimized by variation over the possible input states and detection schemes. The optimization is performed according to the following criteria: (i) maximization of the probability of revealing a perturbation when it actually occurs; (ii) minimization of the smallest perturbation that can be effectively detected.

We look at interferometers as binary communication systems [1, 2], with the perturbation playing the role of the encoded information. According to this view, a general interferometric scheme consists of a source which prepares a state ϱ_0, an intermediate apparatus which may or may not act as a perturbation, and a detector described by a generic POVM Π. The perturbation is described by a unitary operator U_λ and the two possible output states are thus given by ϱ_0, if no perturbation occurs, and $\varrho_\lambda = U_\lambda \varrho_0 U_\lambda^\dagger$, in the case of perturbation. Depending on the outcome of the measurement, one decides on the most probable hypothesis as regards the state of the system. Interferometry is thus equivalent to a binary decision problem, and the corresponding POVM is binary, i.e. there are two possible outcomes.

In this paper we wish to emphasize the role of entanglement in improving interferometric measurements. In particular, we show that an optimized two-mode interferometer requires an entangled input state.

In order to optimize the detection strategies, and to show the benefits of entanglement, we will make use of results and methods from quantum detection theory applied to binary decisions [3]. This approach is particularly useful for our purposes, since it does not refer to any specific detection scheme for the final stage of the interferometer, but rather, owing to its generality, it allows us to find the ultimate quantum limits to interferometry for specific classes of quantum signals [4].

In the next section, in order to establish notation, we briefly review the Neyman–Pearson (NP) approach to quantum binary decisions, and state a lemma regarding minimum input–output overlap. Then, in section 3 we apply these results to the interferometric detection of perturbations and analyse a concrete set-up, based on parametric sources of entanglement and photodetection, that approaches the ultimate bounds on precision. Finally, in section 4 we close the paper with some concluding remarks.

2. Quantum binary decisions in the Neyman–Pearson approach

A measurement aimed at discriminating between two states of a system is described by a binary POVM:

$$\Pi_0, \Pi_\lambda \geqslant 0 \qquad \Pi_0 + \Pi_\lambda = \mathbb{I}. \tag{1}$$

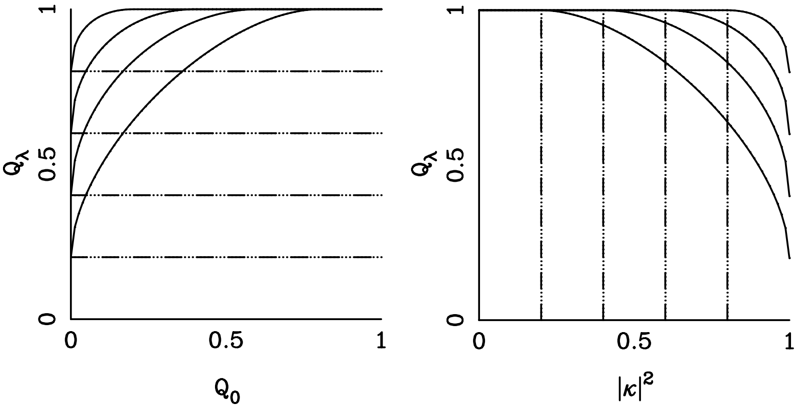

Figure 1. Left: the detection probability Q_λ as a function of the false-alarm probability Q_0 for $|\kappa|^2 = 0.2, 0.4, 0.6, 0.8$ respectively (from top to bottom; the value for $Q_0 = 0$ is $Q_\lambda = 1 - |\kappa|^2$). Right: the detection probability Q_λ as a function of the overlap $|\kappa|^2$ for $Q_0 = 0.2, 0.4, 0.6, 0.8$ respectively (from bottom to top; the value for $|\kappa|^2 = 1$ is $Q_\lambda = Q_0$).

If ϱ_0 and ϱ_λ are orthogonal, i.e. $\varrho_0\varrho_\lambda = \varrho_\lambda\varrho_0 = 0$, the discrimination is trivial, since the POVM in which Π_0 is the projection onto any subspace that contains the support of ϱ_0 and is orthogonal to the support of ϱ_λ perfectly distinguishes the two states. In most cases of interest, however, the states are not orthogonal and one has to apply an optimization scheme. Since interferometric schemes are frequently used for detecting low-rate events, we look for a strategy that has a tolerable probability of wrong inference of perturbation, while leading to a high probability of detecting the perturbation when it actually occurs. For this reason we adopt the so-called NP detection strategy, which consists in fixing the false-alarm probability Q_0—the probability of inferring that the state of the system is ϱ_λ while it is actually ϱ_0—and then maximizing the detection probability Q_λ, i.e. the probability of a correct inference of the state ϱ_λ [5]. It has been proved [3] that this problem can be solved by diagonalizing the operator $\varrho_\lambda - \mu\varrho_0$, $\mu \in \mathbb{R}$ playing the role of a Lagrange multiplier accounting for the bound of fixed false-alarm probability. According to [3], the optimal POVM is the one in which Π_λ is the projection onto the eigenspaces of $\varrho_\lambda - \mu\varrho_0$ relative to positive eigenvalues and $\Pi_0 = \mathbb{I} - \Pi_\lambda$. If ϱ_0 and ϱ_λ are mixed states, such diagonalization is generally not easy. However, when $\varrho_0 = |\psi_0\rangle\langle\psi_0|$ and $\varrho_\lambda = |\psi_\lambda\rangle\langle\psi_\lambda|$ are pure states, it can be easily solved analytically, by expanding $|\psi_0\rangle$ and $|\psi_\lambda\rangle$ in the eigenvectors of the difference operator. In this way one can evaluate both Q_0 and Q_λ versus μ, and after eliminating μ from their expressions one obtains

$$Q_\lambda = \begin{cases} \left[\sqrt{Q_0|\kappa|^2} + \sqrt{(1-Q_0)(1-|\kappa|^2)}\right]^2 & \\ \qquad \text{for } 0 \leqslant Q_0 \leqslant |\kappa|^2, & \\ 1 \qquad \text{for } |\kappa|^2 < Q_0 \leqslant 1 & \end{cases} \tag{2}$$

where $|\kappa|^2 = |\langle\psi_0|\psi_\lambda\rangle|^2 = |\langle\psi_0|U_\lambda|\psi_0\rangle|^2$ is the overlap between the two states. The behaviour of the detection probability is shown in figure 1. According to (2), the smaller the overlap, the easier the discrimination. In fact, if the overlap is smaller than Q_0, one has $Q_\lambda = 1$, while if the overlap is greater than Q_0, Q_λ is a decreasing function of $|\kappa|^2$. In contrast, when the overlap approaches 1, one is forced to decrease the detection probability in order to keep the false-alarm probability small.

After the optimal POVM has been determined, i.e. the optimal detection stage, the whole set-up can be further optimized in looking for the best input state—that is, the state for which $|\kappa|$ assumes its minimum value $|\kappa|_{min}$.

The value $|\kappa|_{min}$ depends on the eigenvalues of the unitary operator U_λ. In order to illustrate this, let us expand U_λ in terms of its eigenvectors $U_\lambda = \sum_j e^{i\varphi_j}|\varphi_j\rangle\langle\varphi_j|$ (with integrals replacing sums in the case of a continuous spectrum) and let us denote by $\mathcal{O}(U_\lambda) = \min_\psi |\langle\psi|U_\lambda|\psi\rangle|^2$ the minimum overlap between the two possible outputs, as obtained by varying the probe state. Then we have the following *overlap lemma* [6, 7]: the minimum overlap $\mathcal{O}(U_\lambda)$ is given by the distance from the origin in the complex plane of the polygon whose vertices are the eigenvalues of U_λ. Therefore, the overlap is either zero (if the polygon includes the origin) or it is given by

$$\mathcal{O}(U_\lambda) = \cos^2\frac{\Delta\varphi}{2}, \tag{3}$$

where $\Delta\varphi$ is the angular spread of the eigenvalues. Moreover, this geometric interpretation of the overlap allows us to calculate the class of optimal states: zero overlap can be achieved with a probe state that is given by a superposition of at least three eigenvectors of U_λ, corresponding to eigenvalues that make a polygon that encloses the origin (or, if they exist, by a superposition of two of them corresponding to diametrically opposed eigenvalues). However, if the minimum overlap is not zero, it is achieved by the optimal probe state given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\varphi_i\rangle + e^{i\theta}|\varphi_j\rangle), \tag{4}$$

with $\Delta\varphi = \varphi_i - \varphi_j$ and θ an arbitrary phase.

3. Interferometry as a binary decision problem

In this section we apply the concepts developed in section 2 to a generic two-mode interferometer, and calculate the optimal input states for the NP ideal strategy. As we will see, the optimal states are entangled. Since they are not physically realizable, we also analyse the ideal scheme with entangled input states that can be concretely prepared and approximate the optimal behaviour. Then we compare the performances of

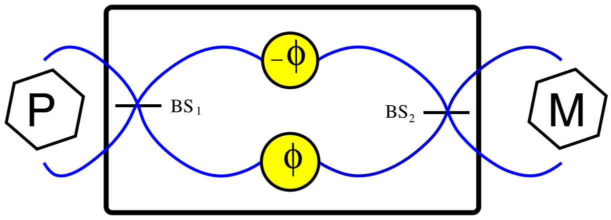

Figure 2. A schematic diagram of a generic two-mode interferometer. The hexagons denote the preparation and the detection stages respectively.

the ideal set-up with that of a realistic Mach–Zehnder scheme based on the entangled twin-beam from a parametric amplifier, and on the measurement of the difference photocurrent at the output. Remarkably, this concrete set-up approaches the performance of the ideal scheme.

The comparison is made in terms of the sensitivity of the interferometer, which is defined as the minimum detectable value λ_{min} of the perturbation parameter λ. More precisely, λ_{min} is meant as the minimum value of λ such that $Q_\lambda/Q_0 = \gamma^* \gg 1/p$, where p is the *a priori* probability of the perturbation. The value of γ^* is fixed by the experimenter and is called the *acceptance ratio*. In order to understand its meaning we note that, if the set-up detects a perturbation, the probability that this inference is true is $P(p,\lambda) = pQ_\lambda/[pQ_\lambda + (1-p)Q_0] = p\gamma^*/[p\gamma^* + (1-p)]$. Therefore, the greater γ^* is, the closer this probability is to one. The condition on λ reads

$$|\kappa|^2 = 1 - \Lambda(Q_0, \gamma^\star)$$
$$\Lambda(Q_0,\gamma^\star) = Q_0\Big[1 + \gamma^\star(1-2Q_0) - 2\sqrt{\gamma^\star(1-Q_0)(1-\gamma^\star Q_0)}\Big], \quad (5)$$

where $|\kappa|^2$ depends parametrically on λ. By solving this equation for the given preparation, one can calculate the sensitivity λ_{min}.

3.1. The NP-optimized interferometer

In figure 2 we show a schematic diagram of a generic two-mode interferometer. The perturbation may act in the surrounded region and, when this occurs, the action on the light beams is described by the unitary operator $V_\phi = \exp\{\mathrm{i}\frac{\phi}{2}(a^\dagger b + ab^\dagger)\}$.

If we consider the Schwinger representation of the Lie algebra of the group $SU(2)$, we have

$$J_+ = a^\dagger b, \qquad J_- = ab^\dagger, \qquad J_z = \tfrac{1}{2}(a^\dagger a - b^\dagger b)$$
$$[J_z, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_z, \quad (6)$$

such that $V_\phi = \exp\{\mathrm{i}\phi J_x\}$ [8]. It is well known that the spectrum of J_x is the set $\mathbb{Z}$ of relative integers, so the spectrum of V_ϕ is the discrete subset $\{\mathrm{e}^{\mathrm{i}m\phi}, m \in \mathbb{Z}\}$ of the unit circle in the complex plane. Apart from the null measure set $\Phi = \{(q/p)\pi, q \in 2\mathbb{Z}+1, p \in \mathbb{Z}\}$ of values of ϕ, the spectrum of V_ϕ is dense in the unit circle and its convex hull contains the origin of the complex plane, although there is no couple of diametrically opposed eigenvalues. If $\phi \in \Phi$, then the optimal state is given by a superposition of two eigenstates of V_ϕ with eigenvalues differing by a factor $\mathrm{e}^{\mathrm{i}\pi}$. In the general case, the optimal state is any superposition of three or more eigenstates of V_ϕ, such that the polygon with vertices on their eigenvalues encloses the origin of the complex plane [7].

Since $J_x = W^\dagger J_z W$ with $W = \exp\{\mathrm{i}\frac{\pi}{2}J_y\}$, the eigenvectors of V_ϕ are entangled. In fact they are obtained from the eigenstates $|n,d\rangle\rangle$ of $a^\dagger a - b^\dagger b$:

$$(a^\dagger a - b^\dagger b)|n,d\rangle\rangle = d|n,d\rangle\rangle,$$
$$\text{where } |n,d\rangle\rangle = \begin{cases} |n+d\rangle \otimes |n\rangle & \text{for } d \geqslant 0, \\ |n\rangle \otimes |n+|d|\rangle & \text{for } d < 0, \end{cases} \quad (7)$$

by the unitary transformation $W^\dagger = \exp\{-\frac{\pi}{4}(a^\dagger b - ab^\dagger)\}$.

In fact, the optimal states are far from being practically realizable. However, we have proved that they are entangled, and this suggests exploring the possibility of performing a reliable discrimination by means of physically realizable entangled states, e.g. twin-beams obtained by a nondegenerate parametric amplifier (NOPA). A twin-beam is a state of the form

$$|x\rangle\rangle = U_g|0\rangle\rangle = \sqrt{1-x^2}\sum_{n=0}^{\infty} x^n |n\rangle \otimes |n\rangle, \quad (8)$$

with $U_g = \exp[g(a^\dagger b^\dagger + ab)]$ (g is the gain of the amplifier), $x = \tanh g$, and with mean photon number $N = 2\sinh^2 g = 2x^2/(1-x^2)$. The overlap for the probe prepared in a twin-beam state is given by

$$\kappa = \langle\langle x|V_\phi|x\rangle\rangle = \langle\langle 0|U_g^\dagger V_\phi U_g|0\rangle\rangle. \quad (9)$$

After some algebra we get

$$|\kappa|^2 = \frac{1}{1 + \frac{4x^2\sin^2\phi}{(1-x^2)^2}} = \frac{1}{1 + N(N+2)\sin^2\phi}. \quad (10)$$

This value is not zero, but it can be arbitrarily small, depending on the mean photon number of the input state. The minimum detectable ϕ, according to (10), is thus given by

$$\phi_{min} = \arcsin\left(\sqrt{\frac{\Lambda(Q_0,\gamma^\star)}{1-\Lambda(Q_0,\gamma^*)}}\frac{1}{\sqrt{N(N+2)}}\right) \simeq \sqrt{\frac{\Lambda(Q_0,\gamma^\star)}{1-\Lambda(Q_0,\gamma^*)}}\frac{1}{N}. \quad (11)$$

3.2. A Mach–Zehnder interferometer fed by a twin-beam

In this section we consider the usual Mach–Zehnder interferometer, i.e. a scheme similar to that of figure 2 but where the detection stage consists of a difference photocurrent measurement (see figure 3). As the input state we consider the entangled twin-beam $|x\rangle\rangle$ produced by a NOPA. The scheme should be feasible with current technology and, as we will see, would approach the ultimate sensitivity bound that has been obtained in the ideal case [4].

After being prepared by the NOPA, the twin-beam enters the interferometer, where it is possibly subjected to the action of the unitary V_ϕ. At the output the two beams are detected and the difference photocurrent $D = a^\dagger a - b^\dagger b$ is measured. If no perturbation occurs, then the output state is still a twin-beam, and since $|x\rangle\rangle$ is an eigenstate of D with zero eigenvalue, we

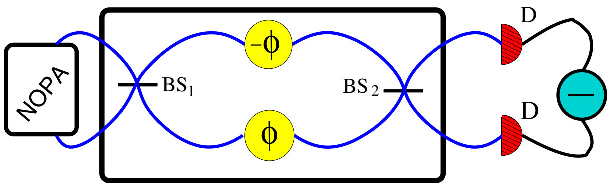

Figure 3. The interferometric scheme used to achieve the ultimate bounds on precision by means of an entangled probe. The NOPA generates a twin-beam which may be subjected to the action of the unitary V_ϕ. At the output the beams are detected and the difference photocurrent is measured. For an unperturbed interferometer the output is again a twin-beam state, and the scheme is designed in order to obtain a constant zero difference photocurrent, whereas a perturbation V_ϕ would produce fluctuations in the difference photocurrent.

have a constant zero outcome for the difference photocurrent. On the other hand, when a perturbation occurs, the output state is no longer an eigenstate of D, and we detect fluctuations which reveal the perturbation.

The false-alarm and the detection probabilities are given by

$$Q_0 = P(d \neq 0|\text{not}\, V_\phi) \equiv 0 \tag{12}$$

$$Q_\lambda = P(d \neq 0|V_\phi) = 1 - P(d \equiv 0|V_\phi), \tag{13}$$

where the probability of observing zero counts at the output, after the action of V_ϕ, is given by

$$P(d \equiv 0|V_\phi) = \sum_n |\langle\langle n, n|V_\phi|x\rangle\rangle|^2, \tag{14}$$

since the eigenvalue $d = 0$ is degenerate. In this case the false-alarm probability is zero and therefore it is not necessary to introduce an acceptance ratio. The scaling of the minimum detectable perturbation can be obtained directly in terms of the detection probability Q_λ using equations (13) and (14):

$$P(d = 0|\phi \neq 0) = 1 - \frac{1}{2}\phi^2 N^2 + \mathrm{O}(\phi^2) \longrightarrow \phi_{min} \simeq \frac{\sqrt{2Q_\lambda}}{N}. \tag{15}$$

One can see that a Mach–Zehnder interferometer fed by a twin-beam shows a sensitivity that scales with the energy as the ideal scheme. The scaling in equation (15) does not depend on any parameter but the energy of the input state. This should be compared with the sensitivity of the customary squeezed states interferometry [9], where the same scaling is achieved only for a very precise tuning of the phase of the squeezing. This means that the entanglement-assisted interferometry provides a more stable and reliable scheme.

It is worth noticing that the experimental measurement of a modulated absorption based on entanglement-assisted difference photocurrent detection has already been performed using an entangled beam exiting an amplifier above threshold (parametric oscillator, OPO) [10].

4. Summary and conclusions

In this paper we analysed two-mode interferometric schemes for the detection of a phase perturbation. We have evaluated the minimum detectable perturbation according to the NP criterion, and have shown that optimal detection schemes need entanglement. Since the NP detection strategy does not correspond to a realistic detector, we suggest the use of a Mach–Zehnder interferometer with a twin-beam as input. Remarkably, the sensitivity of this set-up has the same energy scaling as the ideal scheme. Moreover, the sensitivity depends only on the energy of the input state, thus showing the stability of the present scheme as compared with customary squeezed states interferometry.

We conclude that the technology of entanglement could be of great help in improving the precision and stability of quantum interferometers.

Acknowledgments

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