A quantum measurement of the spin direction

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Abstract

We give a first physical model for the quantum measurement of the spin direction. It is an Arthurs–Kelly model that involves a kind of magnetic–dipole interaction of the spin with three modes of radiation. We show that in a limit of infinite squeezing of radiation the optimal POVM for the measurement of the spin direction is achieved for spin-1/2. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

In quantum mechanics everybody knows that we can measure the component of a spin along a given direction—usually a magnetic field. Why is not possible to measure the "direction" of the spin itself, like what we do in classical mechanics? There must be a way to define the measurement of the direction of the spin also in quantum mechanics, otherwise we would face a situation which is inconsistent with what we normally do in the macroscopic world!

The situation is somewhat similar to what happens for the measurement of position and momentum: in classical mechanics we can measure both jointly, whereas in quantum mechanics we learn that we can measure either one or the other. This is true since only exact (orthogonal) measurements are usually considered. However, from a number of authors [1–4] we also learned that approximate measurements can be considered as well, and in this way we can define joint measurements of position and momentum and, in principle, measurements of the direction of a spin.

The first scheme for a joint measurement of non-commuting observables was introduced by Arthurs and Kelly [5]. The problem of evaluating the minimum added noise in the joint measurement of position and momentum—and, more generally, of a pair of observables whose commutator is not a c-number—was studied in Refs. [6–8]. In the case of two quadratures of one mode of the electromagnetic field the problem can be phrased in

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terms of a coherent positive operator-valued measure (POVM) whose Naimark extension introduces an additional mode of the field: this kind of measurement can be realized by means of a heterodyne detector [9].

The case of a joint measurement of the angular momentum is by far more difficult than the joint measurement of position and momentum. Some optical [10] and Stern–Gerlach [11] measurement schemes have been proposed for discrete-spectrum POVMs for the joint measurement of non-commuting components of the angular momentum. A model based on quantum cloning for the joint measurement of the three components has been studied in Ref. [12].

Spin coherent states have been introduced [13] that can be interpreted as continuous overcomplete POVMs [3, 14], but no physical model has been given for it. It has been shown [15] that such coherent-spin POVM minimizes suitably defined quantities that represent the precision and disturbance of the measurement, and that such POVM would also be optimal for estimating the rotation parameters of a spin [3].

A model for the realization of the measurement of the spin direction would be essential in connecting the quantum with the classical meaning of the angular momentum itself, whence it is of great interest to find viable physical schemes to realize such measurement. The realization of the spin-direction measurement would also represent a first step toward the achievement of the POVM that is needed for teleporting an angular momentum [16].

In this Letter a first physical model for the measurement of the spin direction is presented. It is based on a Arthurs–Kelly model that involves a sort of magnetic-dipole interaction of the spin with three modes of radiation. As we will see, the model achieves the optimal POVM for spin-1/2, in a limit of infinite squeezing of the radiation modes.

2. Coherent-spin POVM

The spin coherent states $|\mathbf{n}\rangle$, with $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, are generated by the action of the unitary operator $V(\mathbf{n}) = \exp[i\theta(\mathbf{n} \wedge \mathbf{k}) \cdot \mathbf{J}]$, $\mathbf{k} = (0, 0, 1)$, on the eigenstate $|-j\rangle$ of J_3 with eigenvalue $-j$. The unitary transformation $V(\mathbf{n})$ is a rotation that maps the direction $\mathbf{k}$ on the direction $\mathbf{n}$, where the rotation is performed continuously along the meridian connecting the two directions. For this reason one has

$$V(\mathbf{n}) J_3 V(\mathbf{n})^\dagger = \mathbf{J} \cdot \mathbf{n}, \tag{1}$$

so that each spin coherent state is the eigenvector of $\mathbf{J} \cdot \mathbf{n}$ with eigenvalue $-j$. This set of states provides the coherent-spin POVM [13] which is given by

$$\Pi(d\mathbf{n}) = \frac{2j+1}{4\pi} |\mathbf{n}\rangle\langle\mathbf{n}| \, d\mathbf{n}. \tag{2}$$

This POVM is a continuous and overcomplete POVM that provides an unbiased estimation of the spin direction. There is no known Naimark extension of POVM (2)—i.e., an orthogonal projective realization of the POVM on an enlarged Hilbert space: we only know that such extension must be infinite-dimensional, since the POVM spectrum is continuous. We emphasize again that coherent-spin POVM (2) minimizes suitably defined precision and disturbance of the measurement [15], and that it is the optimal one for estimating the rotation angles of a spin [3].

3. The Arthurs–Kelly model

In the Arthurs–Kelly scheme [5] the joint measurement of two non-commuting observables on a single quantum system was obtained upon introducing two auxiliary meters. Here we let a spin j interact with three independent radiation modes a_1, a_2, a_3 for a time interval t, according to the following kind of magnetic–dipole interaction:

$$U = \exp\big[-it(J_1Y_1 + J_2Y_2 + J_3Y_3)\big] = \exp[-it\mathbf{J} \cdot \mathbf{Y}], \tag{3}$$

where Y_h denotes the quadrature $Y_h = (i/2)(a_h^\dagger - a_h)$, and all phases have been included in the definition of the annihilation operators a_h. We fix the preparation of the radiation field in the squeezed state $|\Psi\rangle = |\psi_\lambda\rangle_1 |\psi_\lambda\rangle_2 |\psi_\lambda\rangle_3$, where

$$_h\langle y|\psi_\lambda\rangle_h = \left(\frac{2\lambda^2}{\pi}\right)^{1/4} e^{-\lambda^2 y^2}, \tag{4}$$

is the wavefunction on the basis of the eigenstates of Y_h. After the interaction, the measurement of the quadratures $X_h = (1/2)(a_h^\dagger + a_h)$ is performed through independent homodyne detection on each mode. The outcome of the measurement is a vector $\mathbf{x} = x\mathbf{n} = x(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, and the resulting POVM for the spin system is given by

$$\Pi_\lambda(d\mathbf{x}) = \mathrm{Tr}_{\mathrm{rad}}\left[\mathbb{1}_s \otimes |\Psi\rangle\langle\Psi|U^\dagger \mathbb{1}_s \otimes |\mathbf{x}\rangle\langle\mathbf{x}|U\right] d\mathbf{x} = \Omega_\lambda^\dagger(\mathbf{x})\Omega_\lambda(\mathbf{x})\,d\mathbf{x}, \tag{5}$$

where $\mathrm{Tr}_{\mathrm{rad}}$ denotes the trace over radiation modes, $\mathbb{1}_s$ is the identity for the spin Hilbert space, and $\Omega_\lambda(\mathbf{x}) = \langle\mathbf{x}|e^{-it\mathbf{J}\cdot\mathbf{Y}}|\Psi\rangle$. By inserting the completeness relation for the operators Y_h in the matrix element $\langle\mathbf{x}|e^{-it\mathbf{J}\cdot\mathbf{Y}}|\Psi\rangle$, $\Omega_\lambda(\mathbf{x})$ can be rewritten as follows:

$$\Omega_\lambda(\mathbf{x}) = \frac{1}{\pi^{3/2}}\left(\frac{2\lambda^2}{\pi}\right)^{3/4} \int d^3\mathbf{y}\, e^{2i\mathbf{x}\cdot\mathbf{y}} e^{-it\mathbf{J}\cdot\mathbf{y}} e^{-\lambda^2|\mathbf{y}|^2}. \tag{6}$$

We are now interested in the limit of Eq. (6) for infinitely squeezed radiation $\lambda \to 0$.

In order to simplify the analysis, we exploit the rotation covariance of the operator $\Omega_\lambda(\mathbf{x})$,

$$\Omega_\lambda(x\mathbf{n}) = V^\dagger(\mathbf{n})\Omega_\lambda(x\mathbf{k})V(\mathbf{n}), \tag{7}$$

along with the fact that $\Omega_\lambda(x\mathbf{n})$ commutes with $\mathbf{J}\cdot\mathbf{n}$, namely it is diagonal on the eigenvectors of $\mathbf{J}\cdot\mathbf{n}$. Therefore, it is enough to evaluate the matrix elements $\langle m|\Omega(x\mathbf{k})|m\rangle$ on eigenstates $|m\rangle$ of J_z. We also know that such matrix element is real, since Ω_λ is self-adjoint. Therefore, we are interested in the following evaluation:

$$\langle m|\Omega_\lambda(x\mathbf{k})|m\rangle$$
$$= \left(\frac{\lambda}{\pi}\right)^{3/2}\left(\frac{2}{\pi}\right)^{3/4} \int d\mathbf{n} \int_0^\infty dy\, y^2 e^{2i\mathbf{k}\cdot\mathbf{n}xy} e^{-\lambda^2 y^2} \langle m|V(\mathbf{n})e^{-itJ_3 y}V^\dagger(\mathbf{n})|m\rangle$$
$$= \left(\frac{\lambda}{\pi}\right)^{3/2}\left(\frac{2}{\pi}\right)^{3/4} \sum_{m'} \int d\mathbf{n}\, |\langle m|V(\mathbf{n})|m'\rangle|^2 \int_0^\infty dy\, y^2 e^{-im'ty+2i\mathbf{k}\cdot\mathbf{n}xy} e^{-\lambda^2 y^2}. \tag{8}$$

The function $|\langle m|V(\mathbf{n})|m'\rangle|^2$, where $\mathbf{n}$ points in the direction (θ, ϕ), does not depend on ϕ. In fact, if $\mathbf{n}'$ is a unit vector pointing in the direction $(\theta, \phi+\delta)$, one has $V(\mathbf{n}') = \exp[i\delta J_3]V(\mathbf{n})\exp[-i\delta J_3]$, and thus $|\langle m|V(\mathbf{n}')|m'\rangle|^2 = |\langle m|V(\mathbf{n})|m'\rangle|^2 \doteq g_m^{m'}(\cos\theta)$. Since $\langle m|\Omega_\lambda(x\mathbf{k})|m\rangle$ is equal to its real part, the last integral in Eq. (8) can be replaced with its real part. Defining $\eta = \cos\theta$, and integrating on ϕ, Eq. (8) becomes

$$\langle m|\Omega_\lambda(x\mathbf{k})|m\rangle$$
$$= 2\pi\left(\frac{\lambda}{\pi}\right)^{3/2}\left(\frac{2}{\pi}\right)^{3/4} \sum_{m'} \int_{-1}^1 d\eta\, g_m^{m'}(\eta)\frac{(-1)}{4x^2}\partial_\eta^2 \frac{1}{2} \int_{-\infty}^\infty dy\, e^{i(2\eta x - m't)y} e^{-\lambda^2 y}$$
$$= \frac{(-1)}{4x^2}\left(\frac{2}{\pi}\right)^{3/4}\lambda^{1/2} \sum_{m'} \int_{-1}^1 d\eta\, g_{m'}^m(\eta)\partial_\eta^2 \exp\left[\frac{-(2\eta x - m't)^2}{4\lambda^2}\right]. \tag{9}$$

Using the partial integration rule $\int g\,\partial^2 f = g\,\partial f - f\,\partial g + \int \partial^2 g\,f$, and keeping the term $g\,\partial f$, which is the only one that survives in the limit $\lambda \to 0$, one finds

$$\langle m|\Omega_\lambda(x\mathbf{k})|m\rangle \overset{\lambda\to 0}{\approx} \frac{1}{4x^2}\left(\frac{2}{\pi}\right)^{3/4}\lambda^{1/2}\sum_{m'}\left|\langle m|V(\mathbf{n})|m'\rangle\right|^2\frac{x}{\lambda^2}$$

$$\times (2x\cos\theta_\mathbf{n} - m't)\exp\left[-\frac{(2x\cos\theta_\mathbf{n} - m't)^2}{4\lambda^2}\right]\Bigg|_{\mathbf{n}=-\mathbf{k}}^{\mathbf{n}=\mathbf{k}}$$

$$= \frac{1}{2}\left(\frac{2}{\pi}\right)^{3/4}\frac{2x - mt}{x\lambda^{3/2}}\exp\left[\frac{-(2x-mt)^2}{4\lambda^2}\right], \tag{10}$$

where we used $|\langle m|V(\pm\mathbf{k})|m'\rangle|^2 = \delta_{m,\pm m'}$. From this result one has

$$\langle m|\Omega_\lambda^\dagger(x\mathbf{k})\Omega_\lambda(x\mathbf{k})|m\rangle \overset{\lambda\to 0}{\approx} \frac{1}{\pi x^2}\left(\frac{1}{\sqrt{2\pi}\lambda} + \frac{\lambda}{4\sqrt{2\pi}}\partial_x^2\right)\exp\left[-\frac{(2x-mt)^2}{2\lambda^2}\right]. \tag{11}$$

Recalling Eqs. (5), (7), and (1), the POVM writes

$$\Pi(d\mathbf{x}) \approx \frac{1}{\pi}\frac{1}{\sqrt{2\pi}\lambda}\exp\left[-\frac{(2x\mathbb{1} - t\mathbf{J}\cdot\mathbf{n})^2}{2\lambda^2}\right]dx\,d\mathbf{n}. \tag{12}$$

Notice that

$$\langle m|\Pi(d\mathbf{x})|m\rangle \to \begin{cases} \frac{1}{\pi}\delta(2x - mt), & \text{if } mt \neq 0, \\ \frac{1}{2\pi}\delta(2x), & \text{if } mt = 0, \end{cases} \tag{13}$$

where the two different results are due to the fact that $x \geqslant 0$, so that in the case of $mt = 0$ only a half of the Gaussian must be considered. Notice also that depending on the chosen sign of t, only a definite sign of m will contribute for $mt \neq 0$. Therefore, in the limit $\lambda \to 0$, Eq. (12) becomes

$$\Pi(d\mathbf{x}) = \frac{1}{2\pi}\sum_{m,\,mt\geqslant 0}\left(1 - \frac{1}{2}\delta_{m,0}\right)\delta\left(x - \frac{mt}{2}\right)|\mathbf{n},m\rangle\langle\mathbf{n},m|\,dx\,d\mathbf{n}. \tag{14}$$

Eq. (14) shows that only outcome vectors $\mathbf{x}$ with $|\mathbf{x}| = mt/2$ have non-vanishing probability, whence we can also easily express the transformation of the spin state due to the measurement

$$\rho \to \frac{I_{dx,d\mathbf{n}}(\rho)}{\text{Tr}[I_{dx,d\mathbf{n}}(\rho)]},$$

$$I_{dx,d\mathbf{n}}(\rho) = \Omega(\mathbf{x})\rho\Omega^\dagger(\mathbf{x})\,dx\,d\mathbf{n} = \frac{1}{2\pi}\sum_{m,\,mt\geqslant 0}\left(1 - \frac{1}{2}\delta_{m,0}\right)\delta\left(x - \frac{mt}{2}\right)\langle\mathbf{n},m|\rho|\mathbf{n},m\rangle|\mathbf{n},m\rangle\langle\mathbf{n},m|\,dx\,d\mathbf{n}. \tag{15}$$

Taking the marginal POVM—i.e., integrating Eq. (14) over x—for $j = 1/2$ one has

$$\Pi'(d\mathbf{n}) = \frac{1}{2\pi}|\mathbf{n},1/2\rangle\langle\mathbf{n},1/2|\,d\mathbf{n}, \tag{16}$$

which is the coherent-spin POVM for spin-1/2. For larger $j \geqslant 1/2$, instead, we get a "mixed" POVM which is not the optimal coherent-spin one.

4. Conclusions

We have presented a first model for a physical realization of the measurement of the spin direction, corresponding to the coherent-spin POVM, which also represents the optimal estimation of spin rotation angles. In our Arthurs–Kelly model, the spin is coupled with three radiation modes that are homodyne detected after the interaction. In the limit of highly squeezed radiation the coherent-spin POVM is achieved only for spin $j = 1/2$. On the present model we have seen which difficulties we need to face in a quantum measurement of the spin direction, and we hope to have opened the route for a concrete experimental scheme to achieve this new kind of quantum measurement.

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