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Quantum Supermaps

(Supermappe Quantistiche)

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Ai miei genitori

La Nature est un temple où de vivants piliers
Laissent parfois sortir de confuses paroles;
L'homme y passe à travers des forêts de symboles
Qui l'observent avec des regards familiers.

Comme de longs échos qui de loin se confondent
Dans une ténébreuse et profonde unité,
Vaste comme la nuit et comme la clarté,
Les parfums, les couleurs et les sons se répondent.

II est des parfums frais comme des chairs d'enfants,
Doux comme les hautbois, verts comme les prairies,
Et d'autres, corrompus, riches et triomphants,

Ayant l'expansion des choses infinies,
Comme l'ambre, le musc, le benjoin et l'encens,
Qui chantent les transports de l'esprit et des sens.

Charles Baudelaire, *Correspondances*, in *Les Fleurs du mal*

Contents

Introduction

In the formalism of non-relativistic Quantum Mechanics, state preparations are described by density operators on a separable Hilbert space, and state evolutions of closed systems are described by unitary transformations. However, local state evolutions of a composite closed system are no more unitary, this meaning that open quantum systems evolve in a different way. Precisely, physical transformations of open quantum systems are known to be described by Quantum Maps, namely linear maps that inject states into states (state-preserving maps), and such that their local application to a bipartite system is still state-preserving. In the 70's, the fundamental work by Jamiołkowski, Choi, Kraus and many others [1, 2, 3] provided a full mathematical characterization of Quantum Maps. In particular, it turned out that the set of Quantum Maps inherits its structure from that of quantum states, mainly as a consequence of the natural request of state preservation.

Later on, in the framework of *Quantum Information* [4], quantum systems started to be considered as information carriers, so that two-level systems (typically, spin-$\frac{1}{2}$ particles, or orthogonal polarization states of light) were regarded as the fundamental bits of quantum information. Under this perspective, then, Quantum Maps represent the logical operations that may be performed on such *qubits*, so that it is natural to think of them as of 'quantum gates', and to arrange them in 'quantum circuits' where quantum wires represent state evolutions of isolated systems. Furthermore, quantum circuits may be seen, in turn, as quantum gates.

Of course, the choice of using Quantum Maps as quantum gates is made with the aim of obtaining the most general and realistic description of transformations that qubits can undergo. This makes theories of quantum information and quantum computation two very complete and powerful theories, with an overwhelming production of fundamental results in the last two decades.

However, the parallelism between the set of Quantum Maps and that of states, *de facto* setting a strict analogy between qubits and their processing, has not received much attention so far. The present work has its roots in this analogy: indeed, the underlying physical intuition is that, since states and their evolutions share most of their mathematical properties, one may consider evolutions as super-states and then introduce super-maps to describe their physical transformations.

Clearly, in terms of quantum circuits, supermaps correspond to maps of gates into gates, so that they are expected to correspond to some quantum circuit of which the input gate is a component: a trivial example is that of the identity supermap, consisting of the circuit made up by the input gate alone, which maps every gate into itself. However, it is not difficult to guess practical situations where it is necessary to study how the behaviour of some circuit varies as one of its component gates is allowed to be variable – which corresponds to the study of the particular supermap mapping the variable gate into the composite quantum circuit.

Evidently, this would be of great relevance to optimization problems: indeed, one of the most straightforward applications of such a formalism is the optimization of the cloning of gates, which will be considered in the present work. Contrarily to that of states, cloning of maps has received very little attention in literature. Nevertheless, it is not unlikely that some particular tasks of Quantum Computation may require such an operation and, on the other hand, the lack of study on this subject, mainly do to the hardness of the problem, represents one reason more to investigate it.

Furthermore, supermaps are expected to give a formal generalization of maps, so that their introduction acquires a theoretical importance as well.

The present work is structured as follows:

In **Chapter 1**, after reviewing the main axioms of Quantum Mechanics, Quantum Maps are axiomatically introduced as a description of physical transformations of Quantum-Mechanical states: in Section 1.1, necessary conditions that must be fulfilled by Quantum Maps are deduced from physical prescriptions. Then, in Section 1.2 these mathematical conditions are analyzed in detail in order to obtain a handy characterization of Quantum Maps. Finally, in Section 1.3 it is proved that, thanks to such a characterization, those conditions are also sufficient for a Quantum Map to represent state evolutions of open systems, and some further physical remarks are made.

In **Chapter 2**, the notion of Quantum Supermaps is introduced as a mathematical tool for the study of transformations of Quantum Maps. Notice that the structure of this Chapter closely recalls that of Chapter 1: indeed, the axiomatization of Quantum Supermaps being presented in Section 2.1 is carried on in strict analogy with that of Quantum Maps (see Section 1.1), and the properties of Quantum Supermaps are investigated in Section 2.2 with a constant regard to analogous features of Quantum Maps (see Section 1.2). Furthermore, Section 2.4 concludes this Chapter with the important study of the relation between the mathematical formalism of Quantum Supermaps and their physical implementation (as for the case of Quantum Maps, see Section 1.3). An exception to the parallel structures of Chapters 1 and 2 is represented by Section 2.3, where covariant supermaps are introduced mainly as a preparatory study for 1-to-2 Unitary Cloning Supermaps, that are presented in Chapter 3.

In **Chapter 3**, the problem of cloning groups of state transformations is introduced as an application of the formalism that was developed in Chapter 2. In Section 3.1 the general case in which the group of unitaries to be cloned is any compact group is investigated: since an ideal cloning is proved to be impossible in the general case, a strategy for the search of an optimal cloner is outlined. In Section 3.2, the particular case of universal cloning (namely, the problem of cloning all unitary state transformations) is solved for qu*d*its using the strategy and the main results that were developed in the preceding Section.

Table of Common Symbols

$\mathcal{H}$, $\mathcal{K}$, etc. .... Hilbert Spaces
$d_{\mathcal{H}}$ .... Dimension of the Hilbert space $\mathcal{H}$
$|\psi\rangle$, $|\phi\rangle$, etc. .... Elements of $\mathcal{H}$
$\mathcal{L}(\mathcal{H}, \mathcal{K})$ .... Hilbert Space of Linear Applications of $\mathcal{H}$ in $\mathcal{K}$
$\mathcal{B}(\mathcal{H})$ .... Algebra of Bounded Operators on $\mathcal{H}$
$\mathcal{T}(\mathcal{H})$ .... Trace-Class Operators on $\mathcal{H}$
A, B, etc. .... Elements of $\mathcal{B}(\mathcal{H})$ or of $\mathcal{T}(\mathcal{H})$
$\underline{A}$, $\underline{B}$, etc. .... Elements of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ or of $\mathcal{T}(\mathcal{H} \otimes \mathcal{K})$
$\mathbb{1}$ .... Identity operator
$\Omega(\mathcal{H})$ .... Convex Set of Trace-Class Positive Linear Operators on $\mathcal{H}$
$\mathrm{N}_{A_{\mathcal{H}}}(\mathcal{H} \otimes \mathcal{K})$ .... Set ot Operators $\underline{A} \in \mathcal{T}(\mathcal{H} \otimes \mathcal{K})$ with $\mathrm{Tr}_{\mathcal{K}}[\underline{A}] = A_{\mathcal{H}}$
$\mathrm{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ .... Set of States on $\mathcal{H}$
ρ, σ, etc. .... Elements of $\mathrm{S}(\mathcal{H})$
$\underline{\rho}$, $\underline{\sigma}$, etc. .... Elements of $\mathrm{S}(\mathcal{H} \otimes \mathcal{K})$
$\mathrm{Tr}_2 = \mathrm{Tr}_{\mathcal{H}_2}$ .... Partial trace on $\mathcal{H}_2$
$\mathscr{C}$, $\mathscr{E}$, etc. .... Maps on operators
$\mathscr{I}$ .... Identity map
$R_{\mathscr{C}}$ ....Choi operator corresponding to the map $\mathscr{C}$
$\mathscr{C}^{\top}$ ....Map $\mathscr{C}$ in the Heisenberg picture
$\mathrm{SP}, \mathrm{P}, \mathrm{CP}, \mathrm{TP}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}})$ .... Set of $\mathrm{SP}, \mathrm{P}, \mathrm{CP}, \mathrm{TP}$ maps of $\mathcal{T}(\mathcal{H}_{\mathrm{in}})$ into $\mathcal{T}(\mathcal{H}_{\mathrm{out}})$
$\mathrm{QC}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}})$ .... Set of Quantum Channels of $\mathcal{T}(\mathcal{H}_{\mathrm{in}})$ into $\mathcal{T}(\mathcal{H}_{\mathrm{out}})$
$\mathbb{S}$, $\mathbb{T}$, etc. .... Maps acting on maps (Supermaps)
$\mathbb{I}$ .... Identity Supermap
$\mathscr{S}_{\mathbb{S}}$ .... Representing map for the supermap $\mathbb{S}$
$R_{\mathbb{S}}$ .... Choi operator for the supermap $\mathbb{S}$
$\mathrm{QCP}, \mathrm{CP}^2, \mathrm{C}^2\mathrm{P}^2, \mathrm{TP}^2(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}; \mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'})$ .... Set of $\mathrm{QCP}, \mathrm{CP}^2$, $\mathrm{C}^2\mathrm{P}^2, \mathrm{TP}^2$ supermaps of $\mathcal{L}(\mathcal{T}(\mathcal{H}_{\mathrm{in}}), \mathcal{T}(\mathcal{H}_{\mathrm{out}}))$ into $\mathcal{L}(\mathcal{T}(\mathcal{H}_{\mathrm{in}'}), \mathcal{T}(\mathcal{H}_{\mathrm{out}'}))$
$\Theta(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$ .... Set of Choi operators $R_{\mathbb{S}}$ corresponding to TP^2 supermaps $\mathbb{S}$

$\mathbf{G}$, $\mathbf{H}$, etc. .... Groups
$\mathrm{Irrep}_{\mathcal{H}}(U)$ .. Set of inequivalent irreducible subrepresentations of $(U,\mathcal{H})$
$\mathcal{H}^{(\mu)}$ .... μ-th invariant submodule of some representation $(U,\mathcal{H})$
$T^{(\mu)}_{j,i}$ .. Isometry between equivalent invariant submodules $\mathcal{H}^{(\mu)}_i$ and $\mathcal{H}^{(\mu)}_j$
$\mathrm{Ext}(S)$ .... Extremal elements of the convex set S
$F_{\mathbb{S}}(U_g)$ .... Fidelity of the cloning supermap S respect to U_g
$\langle F_{\mathbb{S}}\rangle_{\mathbf{G}}$ .... Mean Fidelity of the cloning supermap S

In the present work, we will also use the symbol $|A\rangle\rangle_{\mathcal{H}\otimes\mathcal{K}}$ to denote the element of $\mathcal{H}\otimes\mathcal{K}$ such that $|A\rangle\rangle_{\mathcal{H}\otimes\mathcal{K}} = (A\otimes\mathbb{1}_{\mathcal{K}})|\mathbb{1}_{\mathcal{K}}\rangle\rangle_{\mathcal{K}^{\otimes 2}} = (\mathbb{1}_{\mathcal{H}}\otimes A^{\top})|\mathbb{1}_{\mathcal{H}}\rangle\rangle_{\mathcal{H}^{\otimes 2}}$, where $A : \mathcal{K} \to \mathcal{H}$ and

$$|\mathbb{1}_{\mathcal{H}}\rangle\rangle = \sum_{i=1}^{d_{\mathcal{H}}} |i\rangle_{\mathcal{H}} \otimes |i\rangle_{\mathcal{H}},$$

where $\{|i\rangle_{\mathcal{H}} \mid i = 1, \ldots, d_{\mathcal{H}}\}$ is any orthonormal basis for $\mathcal{H}$.

Chapter 1

Quantum Maps

In the present Chapter, after reviewing the main axioms of Quantum Mechanics, Quantum Maps are axiomatically introduced as a description of physical transformations of Quantum-Mechanical states: in Section 1.1, necessary conditions that must be fulfilled by Quantum Maps are deduced from physical prescriptions. Then, in Section 1.2 these mathematical conditions are analyzed in detail in order to obtain a handy characterization of Quantum Maps. Finally, in Section 1.3 it is proved that, thanks to such a characterization, those conditions are also sufficient for a Quantum Map to represent open systems' state evolution, and some further physical remarks are made.

1.1 Transformations of Quantum States

In the following, we will use the generic term 'Quantum Map' to describe mathematical superoperators on the set of states describing physical transformations of density operators. The notion of 'physical transformations' will not be uniquely given here: on the contrary, it will be induced by physical considerations during the course of the present treatment. This particular choice of exposition is made with the primary aim to make the axiomatization of supermaps, in Chapter 2, as straightforward as possible.

1.1.1 A brief Review of Quantum Mechanics

In what has come to be known as the standard axiomatization of Quantum Mechanics axiomatization (see, for example, [4]), to each quantum system there corresponds a separable Hilbert space $\mathcal{H}$, to every ensemble of identically prepared systems there corresponds a unit vector $|\psi\rangle \in \mathcal{H}$ (a 'pure state'), and composite systems correspond to the tensor product between all the Hilbert spaces of the component systems[1].

In the framework of Quantum Computation, it is a natural choice to consider physical systems corresponding to finite-dimensional Hilbert spaces only, so that the quantity of information they carry remains finite: any finite-dimensional Hilbert space $\mathcal{H}$ is isomorphic to $\mathbb{C}^d$, where d is the dimension of $\mathcal{H}$: in symbols, we will write $\mathcal{H} \cong \mathbb{C}^d$.

State evolutions of closed systems are assumed to be described by unitary transformations $U \in \mathcal{B}(\mathcal{H})$, that may be represented diagramatically as

$$|\psi_{\text{in}}\rangle \ \xrightarrow{\mathcal{H}_{\text{in}}} \boxed{U} \xrightarrow{\mathcal{H}_{\text{out}}} \ |\psi_{\text{out}}\rangle = U|\psi_{\text{in}}\rangle,$$

where formal labels 'in' and 'out' where introduced for clarity reasons and, of course, $\mathcal{H}_{\text{in}} \cong \mathcal{H}_{\text{out}} \cong \mathcal{H}$.

Ensembles of pure states $\{(|\psi_i\rangle, p_i) \mid i \in I\}$, corresponding to a fraction p_i of systems being prepared in the pure state $|\psi_i\rangle$, are represented by the so-called density operator $\rho \in \mathcal{T}(\mathcal{H})$ ('mixed state'), which is explicitly built as

$$\rho = \sum_{i \in I} p_i |\psi_i\rangle\langle\psi_i|. \tag{1.1}$$

It is usual to discard adjectives 'pure' and 'mixed', since the density operator formalism does not exclude pure states, but rather generalizes them.

It is straightforward to realize that, for any operator $\rho \in \mathcal{T}(\mathcal{H})$ to be interpreted as a density operator, the two following joint conditions are necessary and sufficient:

$$\begin{cases} \rho \geq 0, \\ \mathrm{Tr}[\rho] = 1. \end{cases} \tag{1.2}$$

Indeed, the first condition is necessary and sufficient for ρ to be in the form (1.1) with $\langle\psi_i|\psi_j\rangle = \delta_{i,j}$ and $p_i \geq 0$, and the latter is required for $\{p_i \mid i \in I\}$ to be interpretable as probabilities.

We will use the symbol $\Omega(\mathcal{H})$ to denote the convex cone of trace-class, positive operators on $\mathcal{H}$, and the symbol $\mathrm{N}_1(\mathcal{H})$ to denote the affine subspace of $\mathcal{T}(\mathcal{H})$ such that its elements are normalized to 1 (read 'N' for 'Normalized'): thus, the full set of quantum states on a Hilbert space $\mathcal{H}$ is given by

[1]The explicit relation between normalized vectors in the Hilbert space $\mathcal{H}$ and states of the quantum-mechanical physical system is governed by the so-called Born rule, which estabilishes the correspondence between one normalized vector and the probabilities of the outcomes of measurements of observables. Thus, in order to give this correspondence explicitly one should get into the details of quantum measurement theory: since we w need this formalism in the present treatment, unless in a very superficial fashion, we will not consider it here. However, as a reference for quantum measurement theory see, for example, [3], or more modern reviews such as [5, 6].

$$\mathrm{S}(\mathcal{H}) = \Omega(\mathcal{H}) \cap \mathrm{N}_1(\mathcal{H}) \subset \mathcal{T}(\mathcal{H}). \tag{1.3}$$

Since the intersection of any two convex sets is a convex set too [7], we see that $\mathrm{S}(\mathcal{H})$ is convex, that is, for any two density operators ρ_0, ρ_1 in $\mathrm{S}(\mathcal{H})$, the convex combination

$$\rho_p \doteq p\rho_1 + (1-p)\rho_0 \tag{1.4}$$

is a density operator in $\mathrm{S}(\mathcal{H})$ as well, for all $p \in [0,1]$.

As a result of the convexity of $\mathrm{S}(\mathcal{H})$, every density operator may be decomposed into a convex combination of extremal elements of $\mathrm{S}(\mathcal{H})$: then, from Eq. (1.1) and its following discussion, we realize that all extremal points in $\mathrm{S}(\mathcal{H})$ correspond to density operators in the form $|\psi_i\rangle\langle\psi_i|$, i.e. to pure states. Furthermore, besides statistical ensembles $\{(|\psi_i\rangle, p_i) \mid i \in I\}$ of pure states (corresponding to convex combinations $\rho = \sum_{i\in I} p_i |\psi_i\rangle\langle\psi_i|$ of extremal points $|\psi_i\rangle\langle\psi_i|$ of $\mathrm{S}(\mathcal{H})$), we are allowed to consider statistical ensembles $\{(\rho_i, p_i) \mid i \in I\}$ of mixed states (corresponding to convex combinations $\rho = \sum_{i\in I} p_i\rho_i$ of generic points ρ_i of $\mathrm{S}(\mathcal{H})$).

Clearly, the physical interpretation of such ensembles remains the same, namely convex combinations of density operators (each corresponding to some randomization of pure states) may be regarded as their randomization. Furthermore, it is evident that every non-extremal density operator admits infinite convex decompositions, i.e. there are infinite ways to regard it as the randomization between density operators.

As stated above, composite systems correspond to the tensor product of the respective Hilbert spaces: for instance, if one has two component systems with Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the state of the composite system will be represented by a certain density operator $\underline{\rho} \in \mathrm{S}(\mathcal{H}\otimes\mathcal{K})$. If $\underline{\rho} = \rho\otimes\sigma$ for some $\rho \in \mathrm{S}(\mathcal{H})$ and $\sigma \in \mathrm{S}(\mathcal{K})$, then we will say that the state is *factorized*: this is a very special case which corresponds to the local measurement outcomes of the two systems being statistically independent from each other. In the presence of classical correlations, the state is said to be *separable*, whilst it is *entangled* when correlations are intrinsically due to Quantum Mechanics.

If we decompose a quantum system into two (or more) subsystems, then the local state of one of the two is given by partially tracing the global state on the other degrees of freedom of the system: for instance, given a system in the state $\underline{\rho}$ with Hilbert space $\mathcal{H}$ admitting $\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{H}_2$, the local state of the subsystem corresponding to space $\mathcal{H}_1$ is given by $\mathrm{Tr}_2[\underline{\rho}]$, with

$$\mathrm{Tr}_2 \doteq \mathrm{Tr}_{\mathcal{H}_2} : \begin{cases} \mathcal{T}(\mathcal{H}_1\otimes\mathcal{H}_2) \to \mathcal{T}(\mathcal{H}_1), \\ \underline{\rho} \mapsto \displaystyle\sum_{n=1}^{d_2} \big(\mathbb{1}_{\mathcal{H}_1} \otimes {}_{\mathcal{H}_2}\langle n|\big)\underline{\rho}\big(\mathbb{1}_{\mathcal{H}_1} \otimes |n\rangle_{\mathcal{H}_2}\big), \end{cases} \tag{1.5}$$

where d_2 is the dimension of $\mathcal{H}_2$, and $\{|n\rangle \mid n = 1, \ldots, d_2\}$ is any orthonormal basis for it. As we may expect, partial trace of factorized states is independent of the state which is being traced over, i.e.

$$\begin{aligned} \mathrm{Tr}_2[\rho \otimes \sigma] &= \mathrm{Tr}[\sigma]\,\rho = \\ &= \rho, \end{aligned} \tag{1.6}$$

but this is no more true when the global state $\underline{\rho}$ is entangled or separable.

1.1.2 State Evolutions of Open Systems

The simplest example of a Quantum Map is given, in a natural way, by unitary transformations. In fact, as we have seen, every $\rho \in \mathrm{S}(\mathcal{H})$ can be interpreted as a statistical ensemble of pure quantum states: thus, its time evolution will represent the same ensemble where all component pure states evolved via the same unitary operator U: clearly, this corresponds to

$$\rho_{\mathrm{in}} \xrightarrow{\ \mathcal{H}_{\mathrm{in}}\ } \boxed{U} \xrightarrow{\ \mathcal{H}_{\mathrm{out}}\ } \rho_{\mathrm{out}} = U \rho_{\mathrm{in}} U^\dagger.$$

Despite being, indeed, the broadest class of linear invertible transformations of $\mathcal{B}(\mathcal{H})$ that take the extremal set of pure states $\mathrm{Ext}(\mathrm{S}(\mathcal{H}))$ into itself, Quantum Maps in the form $U \bullet U^\dagger$ (that we can call unitary transformations) are, by far, a very particular subclass of all the possible Quantum Maps $\mathscr{C}$ mapping the *full* set $\mathrm{S}(\mathcal{H})$ into itself.

A priori, in fact, we could merely consider all Quantum Maps $\mathscr{C}$ defined on $\mathrm{S}(\mathcal{H})$ with range contained in $\mathrm{S}(\mathcal{H})$: this is equivalent to state the following

Axiom 1.1 *All Quantum Maps must preserve Quantum States.*

Such a trivial assumption is widely self-explanatory: nevertheless, even though we could have omitted it without any loss of clarity, we consider a good idea to stress all physical requirements we will be making for Quantum Maps, since they will represent some leading steps in the axiomatization of Quantum Supermaps in Chapter 2.

Actually, we realize that there is one further mathematical property we must require at once:

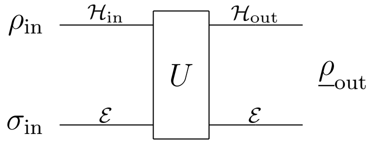
Axiom 1.2 *All Quantum Maps must be convex-linear on the set of states.*

Indeed, let $\rho = \sum_{i \in I} p_i \rho_i$ represent the ensemble of mixed states $\{(\rho_i, p_i) \mid i \in I\}$, and let us consider the transformation $\rho \mapsto \mathscr{C}(\rho)$. Then, the hypothesis of convex-linearity is necessary (and sufficient) to say that $\mathscr{C}(\rho)$ corresponds to the ensemble $\{(\mathscr{C}(\rho_i), p_i) \mid i \in I\}$, namely the initial ensemble where all component states have evolved according to the same transformation.

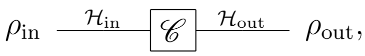
Of course, given a convex-linear Quantum Map $\mathscr{C} : \mathrm{S}(\mathcal{H}) \to \mathscr{C}(\mathrm{S}(\mathcal{H})) \subseteq \mathrm{S}(\mathcal{H})$, we can extend its action to the whole $\mathcal{T}(\mathcal{H})$ by linearity[2]: thus there is no loss of generality in considering linear maps of the whole $\mathcal{T}(\mathcal{H})$, as long as $\mathscr{C}(\mathrm{S}(\mathcal{H})) \subseteq \mathrm{S}(\mathcal{H})$. Furthermore, for this condition to be true, the Quantum Map $\mathscr{C}$ needs to to preserve conditions (1.2) *jointly*, and of course $\mathscr{C} = U \bullet U^\dagger$ represents a very particular kind of Quantum Map.

But, then, under which physical conditions is the Quantum Map allowed to be a $\mathrm{S}(\mathcal{H})$-preserving, linear map different from a unitary transformation? The answer is: when the quantum-statistical system is open.

Indeed, since it is always possible to consider an open system as part of an 'enlarged' closed system, let us consider an environment $\mathcal{E}$ coupled to our open system $\mathcal{H}$, such that the composite system $\mathcal{H} \otimes \mathcal{E}$ may be regarded as closed. Now, even though the initial state of the composite system is separable (say, $\underline{\rho}_{\text{in}} = \rho_{\text{in}} \otimes \sigma_{\text{in}}$), the final state after a time evolution



will be entangled in general, for such is the emerging state $\underline{\rho}_{\text{out}} = U(\rho_{\text{in}} \otimes \sigma_{\text{in}})U^\dagger$. Furthermore, from a local point of view, the open system $\mathcal{H}$ has undergone the transformation



where the output state is given explicitly by

$$\mathscr{C}(\rho_{\text{in}}) = \mathrm{Tr}_{\mathcal{E}}[U(\rho_{\text{in}} \otimes \sigma_{\text{in}})U^\dagger] \tag{1.7}$$

and, of course, $\mathscr{C} : \mathcal{T}(\mathcal{H}_{\text{in}}) \to \mathcal{T}(\mathcal{H}_{\text{out}})$ is a $\mathrm{S}(\mathcal{H})$-preserving linear transformation generally different from a unitary.

Remark 1.1 Notice that, so far, labels 'in' and 'out' could be discharged without any loss of generality nor information: in all cases, the isomorphism $\mathcal{H}_{\text{in}} \cong \mathcal{H}_{\text{out}}$ just looked like a plausible, if not obvious hypothesis. However, this is no more true when we consider open systems. In fact, consider the case in which the state evolution is due to a cloning device for photons: then, we

[2] In fact, let us denote with $\tilde{\mathscr{C}}$ the map defined by $\tilde{\mathscr{C}}(\sum_n \alpha_n \rho_n) \doteq \sum_n \alpha_n \mathscr{C}(\rho_n)$ for all $\{\alpha_n \mid \alpha_n \in \mathbb{C}\}$, for all $\{\rho_n \mid \rho_n \in \mathrm{S}(\mathcal{H})\}$. Then, it is straightforward to realize that $\tilde{\mathscr{C}}$ is a linear map defined on the whole $\mathcal{T}(\mathcal{H})$, and that it is an extension of $\mathscr{C}$.

expect $\mathcal{H}_{\text{out}}$ (several emerging photons) to be larger than $\mathcal{H}_{\text{in}}$ (single input photon). Of course, in order to consider the closure of the system, one shall consider some closed physical system minus the input photon as the initial environmental space $\mathcal{E}_{\text{in}}$, and the same closed physical system minus the output photons as the final environmental space $\mathcal{E}_{\text{out}}$, so that unitary operators of $\mathcal{H}_{\text{in}} \otimes \mathcal{E}_{\text{in}}$ into $\mathcal{H}_{\text{out}} \otimes \mathcal{E}_{\text{out}}$ can be considered. ▲

As a consequence of Remark 1.1, we must replace the notion of $\mathrm{S}(\mathcal{H})$-preserving maps with that of State-Preserving (SP) maps:

Definition 1.1 (SP Map) *Let $\mathscr{C}$ be a linear map of $\mathcal{T}(\mathcal{H}_{\text{in}})$ into $\mathcal{T}(\mathcal{H}_{\text{out}})$: we will say that it is* State-Preserving, *or* SP, *when*

$$\mathscr{C}\big(\mathrm{S}(\mathcal{H}_{\text{in}})\big) \subseteq \mathrm{S}(\mathcal{H}_{\text{out}}), \tag{1.8}$$

and we will denote with $\mathrm{SP}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}})$ *the set of* SP *maps of $\mathcal{T}(\mathcal{H}_{\text{in}})$ into $\mathcal{T}(\mathcal{H}_{\text{out}})$.*

For a review on the theory of open quantum systems see, for example, [8]. In the present treatment we will not need to get into the details of such a theory, as our approach is highly axiomatical.

1.1.3 On Positive and Trace-Preserving Maps

From the definition (1.3) of $\mathrm{S}(\mathcal{H})$, we realize that a *sufficient* condition for $\mathscr{C}$ to be SP is that it maps $\Omega(\mathcal{H}_{\text{in}})$ and $\mathrm{N}_1(\mathcal{H}_{\text{in}})$ respectively into some subset of $\Omega(\mathcal{H}_{\text{out}})$ and of $\mathrm{N}_1(\mathcal{H}_{\text{out}})$. This suggests the two following definitions:

Definition 1.2 (Positive Map) *We will say that the linear map $\mathscr{C} : \mathcal{T}(\mathcal{H}_{\text{in}}) \to \mathcal{T}(\mathcal{H}_{\text{out}})$ is* Positive, *or* P, *when it preserves the positivity of operators, namely when*

$$\mathscr{C}\big(\Omega(\mathcal{H}_{\text{in}})\big) \subseteq \Omega(\mathcal{H}_{\text{out}}), \tag{1.9}$$

and we will denote with $\mathrm{P}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}})$ *the set of* P *maps of $\mathcal{T}(\mathcal{H}_{\text{in}})$ into $\mathcal{T}(\mathcal{H}_{\text{out}})$.*

Definition 1.3 (Trace-Preserving Map) *We will say that the linear map $\mathscr{C} : \mathcal{T}(\mathcal{H}_{\text{in}}) \to \mathcal{T}(\mathcal{H}_{\text{out}})$ is* Trace-Preserving, *or* TP, *when it preserves the normalization of states, namely when*

$$\mathscr{C}\big(\mathrm{N}_1(\mathcal{H}_{\text{in}})\big) \subseteq \mathrm{N}_1(\mathcal{H}_{\text{out}}), \tag{1.10}$$

and we will denote with $\mathrm{TP}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}})$ *the set of* TP *maps of $\mathcal{T}(\mathcal{H}_{\text{in}})$ into $\mathcal{T}(\mathcal{H}_{\text{out}})$.*

Remark 1.2 Thanks to the linearity of $\mathscr{C}$ (and to that of the trace), we can prove that $\mathscr{C}$ is TP in the sense of Definition 1.3 if and only if

$$\mathrm{Tr}[\mathscr{C}(A)] = \mathrm{Tr}[A] \quad \forall\, A \in \mathcal{T}(\mathcal{H}_{\mathrm{in}}). \tag{1.11}$$

Of course, this is a sufficient condition for $\mathscr{C}$ to be TP. To prove that it is also necessary, let $\mathscr{C}$ be TP and let A be any operator with $\mathrm{Tr}[A] \in \mathbb{C} \setminus \{0\}$. Then, we have that $(\mathrm{Tr}[A])^{-1}A$ is an operator with unit trace, so that $\mathscr{C}((\mathrm{Tr}[A])^{-1}A)$ has unit trace too. But since of course $\mathscr{C}((\mathrm{Tr}[A])^{-1}A) = (\mathrm{Tr}[A])^{-1}\mathscr{C}(A)$, then we have that $\mathscr{C}(A)$ has trace equal to $\mathrm{Tr}[A]$. ▲

Lemma 1.1 (On P&TP Maps) *The set of* SP *maps coincides with that of* P&TP *maps. In symbols,*

$$\mathrm{SP}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}) = [\mathrm{P} \cap \mathrm{TP}](\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}). \tag{1.12}$$

Proof As we have alredy noticed, all P&TP maps are trivially SP as well. Then, we have to prove that all SP maps are both P and TP: let us start by showing that all SP maps are P.

Actually, this may be seen as a direct consequence of the trivial fact that the convex cone of positive operators may be parametrized by the (real, positive) trace of its elements:

$$\Omega(\mathcal{H}) = \bigcup_{t \in \mathbb{R}^+} [\Omega(\mathcal{H}) \cap \mathrm{N}_t(\mathcal{H})] \tag{1.13}$$

for all Hilbert spaces $\mathcal{H}$, where we have denoted with $\mathrm{N}_t(\mathcal{H})$ the set of operators on $\mathcal{H}$ that are normalized to (i.e. with trace equal to) t.

Indeed, if $A \in \Omega(\mathcal{H}_{\mathrm{in}}) \cap \mathrm{N}_1(\mathcal{H}_{\mathrm{in}})$, then of course $\mathscr{C}(A) \in \Omega(\mathcal{H}_{\mathrm{out}}) \cap \mathrm{N}_1(\mathcal{H}_{\mathrm{out}})$, as $\mathscr{C}$ is SP by hypothesis. Furthermore, even though A is a nonnormalized positive operator, i.e. $A \in \Omega(\mathcal{H}_{\mathrm{in}}) \setminus \mathrm{N}_1(\mathcal{H}_{\mathrm{in}})$, we still have that $A \in \Omega(\mathcal{H}_{\mathrm{in}}) \cap \mathrm{N}_{\mathrm{Tr}[A]}(\mathcal{H}_{\mathrm{in}})$, with $\mathrm{Tr}[A] \in \mathbb{R}^+ \setminus \{1\}$: thus, apart from the trivial case in which $A = 0$, we have that $(\mathrm{Tr}[A])^{-1}A \in \mathrm{S}(\mathcal{H}_{\mathrm{in}})$ and $\mathscr{C}((\mathrm{Tr}[A])^{-1}A) \in \mathrm{S}(\mathcal{H}_{\mathrm{out}})$. This shows that $\mathscr{C}(A) \in \Omega(\mathcal{H}_{\mathrm{out}}) \cap \mathrm{N}_t(\mathcal{H}_{\mathrm{out}})$ for all $A \in \Omega(\mathcal{H}_{\mathrm{in}}) \cap \mathrm{N}_t(\mathcal{H}_{\mathrm{in}})$ with $t \in \mathbb{R}^+$, i.e.

$$\mathscr{C}\big(\Omega(\mathcal{H}_{\mathrm{in}}) \cap \mathrm{N}_t(\mathcal{H}_{\mathrm{in}})\big) \subseteq \Omega(\mathcal{H}_{\mathrm{out}}) \cap \mathrm{N}_t(\mathcal{H}_{\mathrm{out}}) \quad \forall\, t \in \mathbb{R}^+. \tag{1.14}$$

Substituting in Eq. (1.13) yields $\mathscr{C}(\Omega(\mathcal{H}_{\mathrm{in}})) \subseteq \Omega(\mathcal{H}_{\mathrm{out}})$, namely $\mathscr{C}$ is P.

Finally, we have to prove that, if $\mathscr{C}$ is SP, then it is TP as well. So, let $\mathscr{C}$ be a SP map, and let us prove that $\mathscr{C}$ is TP by contradiction: thus, let us suppose that there exists an operator $A \in \mathrm{N}_1(\mathcal{H}_{\mathrm{in}}) \setminus \Omega(\mathcal{H}_{\mathrm{in}})$ such that

$\mathscr{C}(A) \notin \mathrm{N}_1(\mathcal{H}_{\mathrm{out}})$. Furthermore, let us consider the line $\{\overline{A}_r \mid r \in \mathbb{R}\}$ in the hyperplane $\mathrm{N}_1(\mathcal{H}_{\mathrm{in}})$ parametrized by

$$\overline{A}_r = rA + (1-r)\rho, \quad r \in \mathbb{R}, \tag{1.15}$$

where ρ is some full-rank non-extremal element of $\mathrm{S}(\mathcal{H}_{\mathrm{in}})$. $\overline{A}_r$ is easily checked to be in $\mathrm{N}_1(\mathcal{H}_{\mathrm{in}})$, for all r; furthermore, we have

$$\begin{aligned} r\mathrm{Tr}[\mathscr{C}(A)] &= \mathrm{Tr}[\mathscr{C}(\overline{A}_r)] - (1-r)\mathrm{Tr}\left[\mathscr{C}(\rho)\right] = \\ &= \mathrm{Tr}[\mathscr{C}(\overline{A}_r)] + r - 1, \end{aligned} \tag{1.16}$$

thanks to the hypothesis that $\mathscr{C}$ is SP. So, thanks to the linearity of $\mathscr{C}$ (and to that of the trace), the hypothesis that $\mathrm{Tr}[\mathscr{C}(A)] \neq 1$ yields

$$\mathrm{Tr}[\mathscr{C}(\overline{A}_r)] \neq 1 \quad \forall\, r \in \mathbb{R} \setminus \{0\}. \tag{1.17}$$

Now, since $\overline{A}_0 = \rho$, and since ρ is full-rank and non-extremal in $\mathrm{S}(\mathcal{H}_{\mathrm{in}})$, we can always find an $\varepsilon > 0$ such that $\overline{A}_r$ is still an element of $\mathrm{S}(\mathcal{H}_{\mathrm{in}})$ for all $r \in [-\varepsilon, +\varepsilon]$: this means that

$$\mathrm{Tr}[\mathscr{C}(\overline{A}_r)] = 1 \quad \forall\, r \in [-\varepsilon, +\varepsilon], \tag{1.18}$$

which contradicts Eq. (1.17). ■

1.1.4 On Completely Positive Maps

Summirazing the above results, we have shown that (once considered Axioms 1.1 and 1.2) we may suppose all Quantum Maps to be (linear) P&TP maps of $\mathcal{T}(\mathcal{H}_{\mathrm{in}})$ into $\mathcal{T}(\mathcal{H}_{\mathrm{out}})$: then, Axioms 1.1 and 1.2 are satisfied by

Proposition 1.3 *All Quantum Maps are* P&TP *(linear) maps.*

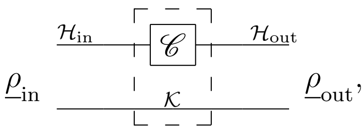
Nevertheless, since we have not given an explicit definition of Quantum Maps (in fact, in the two Axioms we have just formulated two *necessary* conditions for a map to be a Quantum Map), we cannot say, conversely, whether all P&TP maps are, indeed, Quantum Maps, i.e. whether all P&TP maps correspond to some physical transformation of the system.

In fact, as is well known, the whole class of P&TP maps contains several maps which necessarily do not represent physical transformations. Indeed, for a Quantum Map to represent some physical state transformation we must state the following

Axiom 1.4 *Let $\mathcal{H}_{\text{in}}$ be an open system which is entangled with some isolated system $\mathcal{K}$. Then, all Quantum Maps $\mathscr{C}$ of $\mathcal{T}(\mathcal{H}_{\text{in}})$ into some $\mathcal{T}(\mathcal{H}_{\text{out}})$ must be such that the joint evolution $\mathscr{C} \otimes \mathscr{I}_{\mathcal{T}(\mathcal{K})}$ of the composite system $\mathcal{H}_{\text{in}} \otimes \mathcal{K}$ corresponds to a Quantum Map as well.*

The reasons for this request are, once again, very clear: indeed, if Axiom 1.4 were not satisfied, we could have physical evolutions of local states that are no more physical when one takes a global viewpoint.

Now, consider the case in which an open system $\mathcal{H}_{\text{in}}$ is entangled with another system $\mathcal{K}$, and suppose that we can induce some state evolution on the first, by means of a convex-linear P&TP map $\mathscr{C}$, while treating the second as being isolated:



where the output state is given by

$$\underline{\rho}_{\text{out}} = [\mathscr{C} \otimes \mathscr{I}_{\mathcal{T}(\mathcal{K})}](\underline{\rho}_{\text{in}}). \tag{1.19}$$

Clearly, for the above diagram to be interpreted as a state evolution, the dashed rectangle must be described by a P&TP map as well. Now, whilst it is easy to show that $\mathscr{C} \otimes \mathscr{I}_{\mathcal{T}(\mathcal{K})}$ is TP for all TP maps $\mathscr{C}$[3] , it is equally easy to show that it does not necessarily have to be P, even though $\mathscr{C}$ is so[4].

In order to get rid of the above physically meaningless P&TP maps, and to comply with Axiom 1.4, we thus need to restrict the class of P maps to those that preserve positivity for all possible extensions of the system: such maps are known in literature as Completely Positive (CP) maps.

[3]Indeed, let $\mathscr{C}$ be TP and let $\underline{A} \in \mathcal{T}(\mathcal{H}_{\text{in}} \otimes \mathcal{K})$, with Schmidt decomposition given by $\underline{A} = \sum_{i,j} B_i \otimes C_j$. Then,

$$\mathrm{Tr}[[\mathscr{C} \otimes \mathscr{I}](\underline{A})] = \sum_{i,j} \mathrm{Tr}[\mathscr{C}(B_i)]\mathrm{Tr}[C_j] = \sum_{i,j} \mathrm{Tr}[B_i]\mathrm{Tr}[C_j] = \sum_{i,j} \mathrm{Tr}[B_i \otimes C_j] = \mathrm{Tr}[\underline{A}],$$

thanks to the linearity of $\mathscr{C}$ and to that of the trace.

[4]A well-known counter-example is provided by the transposing map $\mathscr{T}$, defined as $\mathscr{T}(\rho) = \rho^\top$, where transposition $\top$ is performed respect to a fixed orthonormal basis $\{|n\rangle\}$ of $\mathcal{H}$: of course this is a P&TP map, for it preserves positivity and normalization conditions (1.2). Nevertheless, consider a bipartite system on $\mathcal{H} \otimes \mathcal{K}$, with $\mathcal{H} \cong \mathcal{K}$, and a transpostion to be performed on the system $\mathcal{H}$ only: then, after defining

$$\begin{cases} |\Psi^-_{mn}\rangle\rangle = \dfrac{1}{\sqrt{2}}\Big[|m\rangle \otimes |n\rangle - |n\rangle \otimes |m\rangle\Big], \\ |\Phi^+_{mn}\rangle\rangle = \dfrac{1}{\sqrt{2}}\Big[|m\rangle \otimes |m\rangle + |n\rangle \otimes |n\rangle\Big], \end{cases} \quad m \neq n,$$

Definition 1.4 (Completely Positive Map) *Any* P *map* $\mathscr{C} : \mathcal{T}(\mathcal{H}_{\rm in}) \to \mathcal{T}(\mathcal{H}_{\rm out})$ *is said to be* Completely Positive*, or* CP*, when every trivial extension of its action on tensor spaces is a* P *map as well, namely when*

$$\mathscr{C} \otimes \mathscr{I}_{\mathcal{T}(\mathcal{K})} \in \mathrm{P}(\mathcal{H}_{\rm in} \otimes \mathcal{K}, \mathcal{H}_{\rm out} \otimes \mathcal{K}) \quad \forall\, \mathcal{K}. \tag{1.20}$$

We will denote with $\mathrm{CP}(\mathcal{H}_{\rm in}, \mathcal{H}_{\rm out})$ *the set of* CP *maps of* $\mathcal{T}(\mathcal{H}_{\rm in})$ *into* $\mathcal{T}(\mathcal{H}_{\rm out})$.

We have shown, thus, that Axioms 1.1, 1.2 and 1.4 lead one to consider Quantum Maps as linear maps of $\mathcal{T}(\mathcal{H}_{\rm in})$ into $\mathcal{T}(\mathcal{H}_{\rm out})$, with the further conditions of Complete Positivity and Trace Preservation. Nevertheless, at this point, we still cannot say whether *all* CP&TP maps may be regarded as Quantum Maps indeed. Luckily enough, the answer to this question is a positive one, as shown by Stinespring [9]: we postpone the details of such an important result to Subsection 1.3.1, and we study the properties of CP and TP maps in the first place.

1.2 Characterization of Quantum Maps

In order to obtain a mathematical characterization of Quantum Maps, in the present Section we study the conditions a Quantum Map must necessarily satisfy.

1.2.1 Positive Maps

Despite being not a sufficient condition for a map to be a Quantum Map, still it may be instructive to study P condition (1.9) briefly. Clearly, such a condition has nothing to do with positivity of $\mathscr{C}$ as a linear operator acting on $\mathcal{T}(\mathcal{H}_{\rm in})$ – which we may write as

$$\mathrm{Tr}[A^\dagger \mathscr{C}(A)] \geq 0 \quad \forall\, A \in \mathcal{T}(\mathcal{H}_{\rm in}). \tag{1.21}$$

In fact, when $\mathcal{H}_{\rm in}$ and $\mathcal{H}_{\rm out}$ are not isomorphic, Eq. (1.21) is not even well defined[5].

---

it is easily seen that, even if the initial state is the pure state $|\Psi^-_{mn}\rangle\rangle$, the final state is no more physical for it is no more positive, as shown by the simple calculation

$$\begin{aligned} \langle\langle \Phi^+_{mn} | \varrho_{\rm out} | \Phi^+_{mn} \rangle\rangle &= \langle\langle \Phi^+_{mn} | \Big[\mathscr{T} \otimes \mathscr{I}_{\mathcal{T}(\mathcal{K})} \Big] \big(|\Psi^-_{mn}\rangle\rangle\langle\langle \Psi^-_{mn}| \big) | \Phi^+_{mn} \rangle\rangle = \\ &= -\frac{1}{2}. \end{aligned}$$

[5] Furthermore, in the case $\mathcal{H}_{\rm in} \cong \mathcal{H}_{\rm out}$, given a positive map $\mathscr{C}$ in the sense of Eq. (1.21), it is easily seen that A being positive is not a sufficient condition for $\mathscr{C}(A)$ to be too. To

Nevertheless, whilst positive maps $\mathscr{C} : \mathcal{T}(\mathcal{H}_{\rm in}) \to \mathcal{T}(\mathcal{H}_{\rm out})$ are by no means related to positive linear operators, it turns out, actually, that they are in $1:1$ correspondence with operators on $\mathcal{H}_{\rm in} \otimes \mathcal{H}_{\rm out}$ that are positive on factorized ones. This is the main result of [1], that we report here as

Theorem 1.2 (Jamiołkowski isomorphism) *Consider two Hilbert spaces, $\mathcal{H}_{\rm in}$ and $\mathcal{H}_{\rm out}$, and let $\{|i\rangle \mid i = 1, \dots, d_{\rm in}\}$ be an orthonormal basis for $\mathcal{H}_{\rm in}$: then, the map*

$$T : \begin{cases} \mathcal{L}(\mathcal{T}(\mathcal{H}_{\rm in}), \mathcal{T}(\mathcal{H}_{\rm out})) \to \mathcal{B}(\mathcal{H}_{\rm in} \otimes \mathcal{H}_{\rm out}), \\ \mathscr{C} \mapsto T_{\mathscr{C}} \doteq \displaystyle\sum_{i,j=1}^{d_{\rm in}} |j\rangle_{\mathcal{H}_{\rm in}}\langle i| \otimes \mathscr{C}\Big(|i\rangle_{\mathcal{H}_{\rm in}}\langle j|\Big), \end{cases} \tag{1.22}$$

estabilishes an isomorphism between linear maps of $\mathcal{T}(\mathcal{H}_{\rm in})$ into $\mathcal{T}(\mathcal{H}_{\rm out})$ and linear operators on $\mathcal{H}_{\rm in} \otimes \mathcal{H}_{\rm out}$ such that P *maps correspond to operators that are positive on factorized states, namely*

$$\begin{gathered} \mathscr{C} \in \mathrm{P}(\mathcal{H}_{\rm in}, \mathcal{H}_{\rm out}) \\ \Updownarrow \\ \langle\psi|\langle\varphi|T_{\mathscr{C}}|\psi\rangle|\varphi\rangle \geq 0 \quad \forall\, |\psi\rangle \in \mathcal{H}_{\rm in}, \quad \forall\, |\varphi\rangle \in \mathcal{H}_{\rm out}. \end{gathered} \tag{1.23}$$

Proof Positivity condition for any map $\mathscr{C} : \mathcal{T}(\mathcal{H}_{\rm in}) \to \mathcal{T}(\mathcal{H}_{\rm out})$ may be written as

$$\langle\varphi|\mathscr{C}\Big(|\psi\rangle\langle\psi|\Big)|\varphi\rangle \geq 0 \quad \forall\, |\psi\rangle \in \mathcal{H}_{\rm in}, \quad \forall\, |\varphi\rangle \in \mathcal{H}_{\rm out}. \tag{1.24}$$

But since

$$\mathscr{C}\Big(|\psi\rangle\langle\psi|\Big) = \sum_{i,j=1}^{d_{\rm in}} \langle\psi|j\rangle\langle i|\psi\rangle\mathscr{C}\Big(|i\rangle\langle j|\Big), \tag{1.25}$$

then condition (1.24) reads

$$\sum_{i,j=1}^{d_{\rm in}} \langle\psi|j\rangle\langle i|\psi\rangle\langle\varphi|\mathscr{C}\Big(|i\rangle\langle j|\Big)|\varphi\rangle \geq 0 \quad \forall\, |\psi\rangle \in \mathcal{H}_{\rm in}, \quad \forall\, |\varphi\rangle \in \mathcal{H}_{\rm out}, \tag{1.26}$$

show this, we build a simple counter-example: let us take $\mathcal{H} \cong \mathbb{C}^2$, and consider the map $\mathscr{C}$ defined as $\mathscr{C}(A) \doteq \frac{1}{2}\mathrm{Tr}[\sigma_z A]\sigma_z$, which is manifestly positive, in the sense of Eq. (1.21), and the density operator $A = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$, with $p \in (1/2, 1]$. The output operator, then, will be $\mathscr{C}(A) = (p - 1/2)\sigma_z$, so that $\langle 0|\mathscr{C}(A)|0\rangle = p - 1/2 < 0$. This proves that, given a positive map $\mathscr{C}$, in the sense of Eq. (1.21), it is not necessarily a P map in the sense of Eq. (1.9). *Vice versa*, given a P map, it does not necesarrily have to be a positive operator: consider again the P transposing map $\mathscr{T}$. Then of course $\mathrm{Tr}[A^\dagger \mathscr{T}(A)] = \mathrm{Tr}[A^\dagger A^\top]$, which is real but not necessarily positive. For example, if $\mathcal{H} \cong \mathbb{C}^2$, then $\mathrm{Tr}[A^\dagger \mathscr{T}(A)] = |\langle 0|A|0\rangle|^2 + |\langle 1|A|1\rangle|^2 + 2\mathbb{R}\mathrm{e}\,[\langle 0|A|1\rangle^*\langle 1|A|0\rangle]$ and it is sufficient to take $A = |0\rangle\langle 1| - |1\rangle\langle 0|$ to show that $\mathrm{Tr}[A^\dagger \mathscr{T}(A)] \leq 0$, despite the fact that $\mathscr{T}$ is P.

which is equivalent to

$$\langle\psi|\otimes\langle\varphi| \left(\sum_{i,j=1}^{d_{\mathrm{in}}} |j\rangle\langle i| \otimes \mathscr{C}\Big(|i\rangle\langle j|\Big) \right) |\psi\rangle\otimes|\varphi\rangle \geq 0 \quad \forall\, |\psi\rangle \in \mathcal{H}_{\mathrm{in}}, \quad \forall\, |\varphi\rangle \in \mathcal{H}_{\mathrm{out}}. \tag{1.27}$$

This completes the proof. ■

Remark 1.3 Condition of positivity on factorized vectors (1.23) is by far weaker than positivity condition on *all* vectors in $\mathcal{H}_{\mathrm{in}} \otimes \mathcal{H}_{\mathrm{out}}$: for instance, it is easily seen that Jamiołkowski operator corresponding to the identical map, $T_{\mathscr{I}}$, is positive on factorized vectors, but it is no more necessarily so on factorizable or entangled ones. ▲

1.2.2 Completely Positive Maps

The most relevant results concerning CP maps were achieved by Choi [2]: the first step for reviewing such results is the well known notion of Choi operators.

Definition 1.5 (Choi operator) *Let $\mathscr{C}$ be any linear map of $\mathcal{T}(\mathcal{H}_{\mathrm{in}})$ into $\mathcal{T}(\mathcal{H}_{\mathrm{out}})$: then, its corresponding* Choi operator $R_{\mathscr{C}} \in \mathcal{B}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$ *is given by*

$$R_{\mathscr{C}} \doteq [\mathscr{C} \otimes \mathscr{I}_{\mathcal{T}(\mathcal{H}_{\mathrm{in}})}]\Big(|\mathbb{1}_{\mathrm{in}}\rangle\rangle\langle\langle\mathbb{1}_{\mathrm{in}}|\Big). \tag{1.28}$$

Remark 1.4 Eq. (1.28) estabilishes an isomorphism between maps and operators explicitly given by

$$R : \begin{cases} \mathcal{L}(\mathcal{T}(\mathcal{H}_{\mathrm{in}}), \mathcal{T}(\mathcal{H}_{\mathrm{out}})) \to \mathcal{B}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}), \\ \mathscr{C} \mapsto R_{\mathscr{C}}. \end{cases} \tag{1.29}$$

Indeed, it is easy to check that the inverse isomorphism is given by

$$R \mapsto \mathscr{C}_R \mid \mathscr{C}_R(A) = \mathrm{Tr}_{\mathrm{in}} \left[\left(\mathbb{1}_{\mathrm{out}} \otimes A^{\top} \right) R \right] \quad \forall\, A \in \mathcal{T}(\mathcal{H}_{\mathrm{in}}). \tag{1.30}$$

A priori, this is not a relevant isomorphism, for it simply states that linear applications between two spaces with dimensions d_{in}^2 and d_{out}^2 may be seen as $d_{\mathrm{out}}^2 \times d_{\mathrm{in}}^2$ matrices. Nevertheless, it acquires a very important meaning when one considers CP maps, as the following Theorem shows. ▲

Theorem 1.3 (Choi isomorphism) *Given a linear map* $\mathscr{C} : \mathcal{T}(\mathcal{H}_{\text{in}}) \to \mathcal{T}(\mathcal{H}_{\text{out}})$*, the following propositions are equivalent:*

1. $\mathscr{C}$ *is* CP*.*
2. *Its Choi operator* $R_{\mathscr{C}}$ *is positive.*
3. *There exists a set* $\{M_x \mid x \in X\} \subset \mathcal{L}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}})$ *such that*

$$\mathscr{C}(A) = \sum_{x\in X} M_x A M_x^\dagger \quad \forall\, A \in \mathcal{T}(\mathcal{H}_{\text{in}}). \tag{1.31}$$

Proof First we note that (1) ⇒ (2) thanks to Definitions 1.4 & 1.5 (of CP maps and Choi operators, respectively), since $|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|$ is a positive operator.

Now, given any linear map $\mathscr{C}$, we let $R_{\mathscr{C}}$ be positive as in prop. (2): we may thus write its diagonalization as

$$R_{\mathscr{C}} = \sum_{x\in X} |M_x\rangle\rangle\langle\langle M_x|, \tag{1.32}$$

for some finite set $\{|M_x\rangle\rangle \mid x \in X\} \subset \mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}$, from which follows

$$\begin{aligned} R_{\mathscr{C}} &= \sum_{x\in X} (M_x \otimes \mathbb{1}_{\mathcal{H}_{\text{in}}})|\mathbb{1}_{\mathcal{H}_{\text{in}}}\rangle\rangle\langle\langle\mathbb{1}_{\mathcal{H}_{\text{in}}}|(M_x^\dagger \otimes \mathbb{1}_{\mathcal{H}_{\text{in}}}) = \\ &= \sum_{i,j=1}^{d_{\text{in}}} \left(\sum_{x\in X} M_x |i\rangle_{\text{in}}\langle j| M_x^\dagger\right) \otimes |i\rangle_{\text{in}}\langle j|. \end{aligned} \tag{1.33}$$

But, since of course Eq. (1.28) may be rephrased as

$$R_{\mathscr{C}} = \sum_{i,j=1}^{d_{\text{in}}} \mathscr{C}\Big(|i\rangle_{\text{in}}\langle j|\Big) \otimes |i\rangle_{\text{in}}\langle j|, \tag{1.34}$$

then we have proved that

$$\mathscr{C}\Big(|i\rangle_{\text{in}}\langle j|\Big) \equiv \sum_{x\in X} M_x |i\rangle_{\text{in}}\langle j| M_x^\dagger, \tag{1.35}$$

i.e. proposition (3).

Finally, consider any positive operator $A \in \mathcal{T}(\mathcal{H}_{\text{in}} \otimes \mathcal{K})$ with diagonalization

$$A = \sum_{y\in Y} |N_y\rangle\rangle\langle\langle N_y|, \tag{1.36}$$

for a proper finite set $\{|N_y\rangle\rangle \mid y \in Y\} \subset \mathcal{H}_{\text{in}} \otimes \mathcal{K}$. Then, for every map $\mathscr{C} : \mathcal{T}(\mathcal{H}_{\text{in}}) \to \mathcal{T}(\mathcal{H}_{\text{out}})$ we have

$$\begin{aligned}
[\mathscr{C} \otimes \mathscr{I}_{\mathcal{T}(\mathcal{K})}](A) = [\mathscr{C} \otimes \mathscr{I}_{\mathcal{T}(\mathcal{K})}] \left(\sum_{y\in Y} |N_y\rangle\rangle_{\mathcal{H}_{\text{in}}\otimes\mathcal{K}} \langle\langle N_y| \right) = \\
= [\mathscr{C} \otimes \mathscr{I}_{\mathcal{T}(\mathcal{K})}] \left(\sum_{y\in Y} (N_y \otimes \mathbb{1}_{\mathcal{K}}) |\mathbb{1}_{\mathcal{K}}\rangle\rangle\langle\langle \mathbb{1}_{\mathcal{K}}| (N_y^\dagger \otimes \mathbb{1}_{\mathcal{K}}) \right) = \\
= [\mathscr{C} \otimes \mathscr{I}_{\mathcal{T}(\mathcal{K})}] \left(\sum_{i,j=1}^{d_{\mathcal{K}}} \sum_{y\in Y} N_y |i\rangle_{\mathcal{K}}\langle j| N_y^\dagger \otimes |i\rangle_{\mathcal{K}}\langle j| \right) = \\
= \sum_{y\in Y} \sum_{i,j=1}^{d_{\mathcal{K}}} \mathscr{C}\Big(N_y |i\rangle_{\mathcal{K}}\langle j| N_y^\dagger \Big) \otimes |i\rangle_{\mathcal{K}}\langle j|,
\end{aligned} \tag{1.37}$$

thanks to the linearity of $\mathscr{C}$. Now, if we choose $\mathscr{C}$ as in prop. (3), then

$$\begin{aligned}
[\mathscr{C} \otimes \mathscr{I}_{\mathcal{T}(\mathcal{K})}](A) &= \sum_{x\in X} \sum_{y\in Y} \sum_{i,j=1}^{d_{\mathcal{K}}} M_x N_y |i\rangle_{\mathcal{K}}\langle j| N_y^\dagger M_x^\dagger \otimes |i\rangle_{\mathcal{K}}\langle j| = \\
&= \sum_{x\in X} \sum_{y\in Y} |M_x N_y\rangle\rangle\langle\langle M_x N_y|,
\end{aligned} \tag{1.38}$$

which is manifestly positive: this proves that (3) $\Rightarrow$ (1). ■

Remark 1.5 In literature, Eq. (1.31) is commonly known as *Kraus decomposition* (or *Operator-Sum Representation*) for the CP map $\mathscr{C}$ [3]: we will also call in the same way the particular set of Kraus operators $\{M_x \mid x \in X\}$. Since operators $\{M_x \mid x \in X\}$ derive from the diagonalization of a positive operator, it is natural to require them to be orthogonal: in which case, we will say that the Kraus decomposition is in its *canonical form*. On the other hand, it is still possible to have non-canonical Kraus decompositions: if operators are still linearly independent, though, we will say that the decomposition is in its *minimal form*. Of course, then, the canonical form is minimal as well.

The problem of finding a characterization for the class of Kraus decompositions, given a CP map $\mathscr{C}$, is once again solved by Choi [2], and the Theorem is reported below, proofless.

Theorem 1.4 (Characterization of Kraus decompositions) *Let $\mathscr{C}$ be a* CP *map, and let $\{M_x \mid x \in X\}$ be a minimal Kraus decomposition for $\mathscr{C}$. Then $\{N_y \mid y \in Y\}$ is another Kraus decomposition for $\mathscr{C}$ if and only if there exists a $|Y| \times |X|$ isometric matrix V such that*

$$N_y = \sum_{x\in X} V_{yx} M_x \quad \forall\, y \in Y. \tag{1.39}$$

Furthermore, if Kraus decomposition $\{N_y \mid y \in Y\}$ *is minimal too, then* $|X| = |Y|$ *and* V *is unitary.*

▲

Remark 1.6 Theorem 1.3 provides two equivalent methods for characterizing CP maps: Choi operators and Kraus decompositions. Whilst choosing between the two might be just a matter of convenience, in most cases the formalism of Choi operators turns out to be quite more compact and straightforward. Nevertheless, Kraus decomposition has the advantage of expliciting the physical interpretation of CP&TP maps: we shall thus study TP maps before making this point. ▲

Remark 1.7 The main result of Theorem 1.3 states that the set of CP maps from $\mathcal{T}(\mathcal{H}_{\text{in}})$ into $\mathcal{T}(\mathcal{H}_{\text{out}})$ is isomorphic to the convex cone of positive operators on $\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}$: this may be written in symbols as

$$\mathrm{CP}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}) \cong \Omega(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}), \tag{1.40}$$

and it has the important consequence that CP maps form a convex set. ▲

1.2.3 Trace-Preserving Maps

Since Choi isomorphism proved to be an excellent tool to deal with the characterization of CP maps, it is natural to try and characterize TP maps in the same manner: fortunately, such characterization is possible, as shown in the following

Lemma 1.5 (Characterization of TP **Maps)** *Let* $\mathscr{C}$ *be* any *linear map of* $\mathcal{T}(\mathcal{H}_{\text{in}})$ *into* $\mathcal{T}(\mathcal{H}_{\text{out}})$*. Then,* $\mathscr{C}$ *is* TP *if and only if its Choi operator* $R_{\mathscr{C}} \in \mathcal{B}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$ *satisfies*

$$\mathrm{Tr}_{\text{out}}[R_{\mathscr{C}}] = \mathbb{1}_{\text{in}}. \tag{1.41}$$

Proof First, we note that

$$\mathrm{Tr}_{\text{out}}[R_{\mathscr{C}}] = \sum_{i,j=1}^{d_{\text{in}}} \mathrm{Tr}\left[\mathscr{C}\Big(|i\rangle_{\text{in}}\langle j|\Big)\right] |i\rangle_{\text{in}}\langle j|, \tag{1.42}$$

where $\{|i\rangle \mid i = 1, \dots d_{\mathrm{in}}\}$ is an orthonormal basis for $\mathcal{H}_{\mathrm{in}}$, so that condition (1.41) is equivalent to

$$\mathrm{Tr}\left[\mathscr{C}\Big(|i\rangle_{\mathrm{in}}\langle j|\Big)\right] = \delta_{i,j}. \tag{1.43}$$

Then, if the last Equation holds, we have that

$$\begin{aligned} \mathrm{Tr}[\mathscr{C}(A)] &= \sum_{i,j=1}^{d_{\mathrm{in}}} \langle i|A|j\rangle \mathrm{Tr}\left[\mathscr{C}\Big(|i\rangle_{\mathrm{in}}\langle j|\Big)\right] = \\ &= \sum_{i=1}^{d_{\mathrm{in}}} \langle i|A|i\rangle = \\ &= \mathrm{Tr}[A] \quad \forall\, A \in \mathcal{T}(\mathcal{H}_{\mathrm{in}}), \end{aligned} \tag{1.44}$$

i.e. $\mathscr{C}$ is TP. *Vice-versa*, if $\mathscr{C}$ is TP, then of course we have that Eq. (1.43) is trivially satisfied. ■

Remark 1.8 Lemma 1.5 tells us that the set of TP maps of $\mathcal{T}(\mathcal{H}_{\mathrm{in}})$ into $\mathcal{T}(\mathcal{H}_{\mathrm{out}})$ is isomorphic to the affine hyperplane (in $\mathcal{B}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$) of Choi operators that are normalized to $\mathbb{1}_{\mathrm{in}}$: in symbols,

$$\mathrm{TP}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}) \cong \mathrm{N}_{\mathbb{1}_{\mathrm{in}}}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}), \tag{1.45}$$

where we put

$$\mathrm{N}_{\mathbb{1}_{\mathrm{in}}}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}) \doteq \{R \in \mathcal{T}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}) \mid \mathrm{Tr}_{\mathrm{out}}[R] = \mathbb{1}_{\mathrm{in}}\}. \tag{1.46}$$

Thus, the set of TP maps is an affine space. ▲

1.2.4 Completely Positive & Trace-Preserving Maps

Theorem 1.3 and Lemma 1.5, respectively, provide us with full characterization of CP and TP maps. The following Corollary provides us with an explicit characterization of maps that are jointly CP&TP.

Corollary 1.6 (to Lemma 1.5) *Any linear map $\mathscr{C}$ of $\mathcal{T}(\mathcal{H}_{\mathrm{in}})$ into $\mathcal{T}(\mathcal{H}_{\mathrm{out}})$ is* CP&TP *if and only if there exists a set $\{M_x \mid x \in X\} \subset \mathcal{L}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}})$ such that*

$$\mathscr{C}(A) = \sum_{x\in X} M_x A M_x^\dagger \quad \forall\, A \in \mathcal{T}(\mathcal{H}_{\mathrm{in}}) \tag{1.47}$$

and

$$\sum_{x\in X} M_x^\dagger M_x = \mathbb{1}_{\mathrm{in}}. \tag{1.48}$$

Proof Of course, $\mathscr{C}$ is CP if and only if Eq. (1.47) holds. So, let $\mathscr{C}$ be CP: from the proof of Theorem 1.3, we know that Kraus decomposition $\{M_x\} \subset \mathcal{L}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}})$ is given by the diagonalization of the positive Choi operator, i.e.

$$R_{\mathscr{C}} = \sum_{x \in X} |M_x\rangle\rangle\langle\langle M_x|, \tag{1.49}$$

for some $\{|M_x\rangle\rangle \mid x \in X\} \subset \mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}$. Then,

$$\begin{aligned}
\mathrm{Tr}_{\text{out}}[R_{\mathscr{C}}] &= \sum_{x \in X} \mathrm{Tr}_{\text{out}}[|M_x\rangle\rangle_{\text{out,in}}\langle\langle M_x|] = \\
&= \sum_{x \in X} \mathrm{Tr}_{\text{out}_1}[(\mathbb{1}_{\text{out}_1} \otimes M_x^\top)|\mathbb{1}\rangle\rangle_{\text{out}_1,\text{out}_2}\langle\langle \mathbb{1}|(\mathbb{1}_{\text{out}_1} \otimes M_x^*)] = \\
&= \sum_{x \in X} M_x^\top \mathrm{Tr}_{\text{out}_1}[|\mathbb{1}\rangle\rangle_{\text{out}_1,\text{out}_2}\langle\langle \mathbb{1}|] M_x^* = \\
&= \sum_{x \in X} M_x^\top M_x^*.
\end{aligned} \tag{1.50}$$

Now, using Lemma 1.5, we obtain

$$\mathscr{C} \in \mathrm{TP}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}) \Leftrightarrow (1.41) \Leftrightarrow \sum_{x \in X} M_x^\top M_x^* = \mathbb{1}_{\text{in}} \Leftrightarrow (1.48). \tag{1.51}$$

This completes the proof. ■

Remark 1.9 We stress the fact that Eq. (1.41) is a necessary and sufficient TP condition for *all* linear maps, whilst Eq. (1.48) is necessary and sufficient only when we consider CP maps along with some Kraus decomposition $\{M_x \mid x \in X\}$. ▲

1.3 Quantum Channels

Now that we have obtained the desired mathematical characterization of CP&TP maps, we proceed to study the relation between such maps and state transformations of quantum systems.

1.3.1 Stinespring Theorem

In the following, we will state a simplified version of Stinespring Theorem [9]: as we pointed out before, this will let us claim that all state evolutions of open systems are represented by CP&TP maps, and also that all CP&TP maps represent some physical evolution. In order to state the main result, we will require the notion of Heisenberg picture of a map.

Definition 1.6 (Heisenberg Picture of a Map) *Let $\mathscr{C}$ be a map of $\mathcal{T}(\mathcal{H})$ into $\mathcal{T}(\mathcal{K})$. We will say that,* in the Heisenberg picture*, the map is represented by $\mathscr{C}^\top : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ such that*

$$\mathrm{Tr}[B\mathscr{C}(A)] = \mathrm{Tr}[\mathscr{C}^\top(B)A] \quad \forall\, (A,B) \in \mathcal{T}(\mathcal{H}) \times \mathcal{B}(\mathcal{K}). \tag{1.52}$$

Remark 1.10 The above condition may be expanded to

$${}_{\mathcal{H}}\langle i|\mathscr{C}^\top\Big(|k\rangle_{\mathcal{K}}\langle l|\Big)|j\rangle_{\mathcal{H}} = {}_{\mathcal{K}}\langle l|\mathscr{C}\Big(|j\rangle_{\mathcal{H}}\langle i|\Big)|k\rangle_{\mathcal{K}}, \tag{1.53}$$

where $\{|i\rangle\}$ and $\{|k\rangle\}$ are orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$, respectively. Then, this is equivalent to

$$\begin{aligned}
&\sum_{i,j=1}^{d_{\mathcal{H}}}\sum_{k,l=1}^{d_{\mathcal{K}}}\Big[{}_{\mathcal{H}}\langle i|\mathscr{C}^\top\Big(|k\rangle_{\mathcal{K}}\langle l|\Big)|j\rangle_{\mathcal{H}}\Big]\,|i\rangle_{\mathcal{H}}\langle j|\otimes|k\rangle_{\mathcal{K}}\langle l| = \\
&= \sum_{i,j=1}^{d_{\mathcal{H}}}\sum_{k,l=1}^{d_{\mathcal{K}}}\Big[{}_{\mathcal{K}}\langle k|\mathscr{C}\Big(|i\rangle_{\mathcal{H}}\langle j|\Big)|l\rangle_{\mathcal{K}}\Big]\,|l\rangle_{\mathcal{K}}\langle k|\otimes|j\rangle_{\mathcal{H}}\langle i|,
\end{aligned} \tag{1.54}$$

which, in turn, may be rewritten as

$$R_{\mathscr{C}^\top} = R_{\mathscr{C}}^\top. \tag{1.55}$$

However, notice that Hilbert spaces are ordered differently on the two sides of Eq. (1.55). ▲

Remark 1.11 Thanks to Remark 10, it is easy to show that we have the two following logical equivalences:

$$\begin{array}{lcccl}
\mathscr{C} \text{ is CP} & \Leftrightarrow & R_{\mathscr{C}^\top} \geq 0 & \Leftrightarrow & \mathscr{C}^\top \text{ is CP}, \\
\mathscr{C} \text{ is TP} & \Leftrightarrow & \mathrm{Tr}_{\mathcal{K}}[R_{\mathscr{C}^\top}]\mathbb{1}_{\mathcal{H}} & \Leftrightarrow & \mathscr{C}^\top \text{ is unital.}
\end{array} \tag{1.56}$$

Whilst the former is trivial, to prove the latter it is sufficient to consider the inverse isomorphism formula (1.30), that we rewrite here for $\mathscr{C}^\top$ as

$$\mathscr{C}^\top(B) = \mathrm{Tr}_{\mathcal{K}}[(\mathbb{1}_{\mathcal{H}} \otimes B^\top)R_{\mathscr{C}^\top}] \quad \forall\, B \in \mathcal{T}(\mathcal{K}). \tag{1.57}$$

Then TP condition is evidently equivalent to $\mathscr{C}^\top(\mathbb{1}_{\mathcal{K}}) = \mathbb{1}_{\mathcal{H}}$.

Furthermore, it is straightforward to check that

$$
\begin{aligned}
R_{\mathscr{C}} &= \sum_x |M_x\rangle\rangle_{\mathcal{K}\otimes\mathcal{H}}\langle\langle M_x| \\
&\Updownarrow \\
R_{\mathscr{C}^\top} &= \sum_x |M_x^\dagger\rangle\rangle_{\mathcal{H}\otimes\mathcal{K}}\langle\langle M_x^\dagger|,
\end{aligned}
\tag{1.58}
$$

namely Kraus operators for $\mathscr{C}^\top$ are the Hermitian conjugates of Kraus operators for $\mathscr{C}$. ▲

We are now ready to state the Theorem.

Theorem 1.7 (Simplified Stinespring Theorem) *Let $\mathscr{C}^\top$ be a map of $\mathcal{B}(\mathcal{H}_{\text{out}})$ into $\mathcal{B}(\mathcal{H}_{\text{in}})$: then, $\mathscr{C}^\top$ is* CP *if and only if*

$$
\mathscr{C}^\top(B) = V^\dagger(B \otimes \mathbb{1}_{\mathcal{E}_{\text{out}}})V \quad \forall\, B \in \mathcal{B}(\mathcal{H}_{\text{out}}), \tag{1.59}
$$

where V is some linear map of $\mathcal{H}_{\text{in}}$ into $\mathcal{H}_{\text{out}} \otimes \mathcal{E}_{\text{out}}$ for some Hilbert space $\mathcal{E}_{\text{out}}$.

Proof Let Eq. (1.59) hold, and let us define the set $\{M_x \mid x = 1, \ldots, d_{\mathcal{E}_{\text{out}}}\}$ of linear transformations of $\mathcal{H}_{\text{in}}$ into $\mathcal{H}_{\text{out}}$ as

$$
M_x \doteq \Big(\mathbb{1}_{\text{out}} \otimes {}_{\mathcal{E}_{\text{out}}}\langle x|\Big)V, \tag{1.60}
$$

where $\{|x\rangle_{\mathcal{E}_{\text{out}}}\}$ is any orthonormal basis for $\mathcal{E}_{\text{out}}$: then, we have

$$
V = \sum_{x=1}^{d_{\mathcal{E}_{\text{out}}}} \Big(\mathbb{1}_{\text{out}} \otimes |x\rangle_{\mathcal{E}_{\text{out}}}\Big)M_x, \tag{1.61}
$$

and substituting in (1.59) yields

$$
\begin{aligned}
\mathscr{C}^\top(B) &= \sum_{x,y=1}^{d_{\mathcal{E}_{\text{out}}}} M_y^\dagger\Big(\mathbb{1}_{\text{out}} \otimes {}_{\mathcal{E}_{\text{out}}}\langle y|\Big)(B \otimes \mathbb{1}_{\mathcal{E}_{\text{out}}})\Big(\mathbb{1}_{\text{out}} \otimes |x\rangle_{\mathcal{E}_{\text{out}}}\Big)M_x = \\
&= \sum_{x=1}^{d_{\mathcal{E}_{\text{out}}}} M_x^\dagger B M_x \quad \forall\, B \in \mathcal{B}(\mathcal{H}_{\text{out}}).
\end{aligned}
\tag{1.62}
$$

Thus, thanks to the Theorem by Choi 1.3, $\mathscr{C}^\top$ is CP. Of course, the proof may be reversed, so that the 'if' becomes an 'iff'. ■

Corollary 1.8 (to Theorem 1.7) *Let $\mathscr{C}$ be a map of $\mathcal{T}(\mathcal{H}_{\rm in})$ into $\mathcal{T}(\mathcal{H}_{\rm out})$. Then, $\mathscr{C}$ is* CP&TP *if and only if it admits the following representation:*

$$\mathscr{C}(A) = \mathrm{Tr}_{\mathcal{E}_{\rm out}}\Big[U_\psi\big(A \otimes |\psi\rangle_{\mathcal{E}_{\rm in}}\langle\psi|\big)U_\psi^\dagger\Big] \quad \forall\, A \in \mathcal{T}(\mathcal{H}_{\rm in}), \tag{1.63}$$

for some Hilbert spaces $\mathcal{E}_{\rm in}$, $\mathcal{E}_{\rm out}$, where $|\psi\rangle$ is any pure state of $\mathcal{E}_{\rm in}$ and $U_\psi : \mathcal{H}_{\rm in} \otimes \mathcal{E}_{\rm in} \to \mathcal{H}_{\rm out} \otimes \mathcal{E}_{\rm out}$ is a unitary transformation depending, other than on $\mathscr{C}$, on the choice of $|\psi\rangle$.

Proof $\mathscr{C}$ is CP if and only $\mathscr{C}^\top$ is so, i.e. if and only if Eq. (1.59) holds. Now, as we have seen, $\mathscr{C}$ is TP if and only if $\mathscr{C}^\top$ is unital, that is $\mathscr{C}^\top(\mathbb{1}_{\rm out}) = \mathbb{1}_{\rm in}$. Then, we have proved that $\mathscr{C}$ is CP&TP if and only if $\mathscr{C}^\top$ satisfies Eq. (1.59), V being an isometry. Furthermore, from the Definition 1.6 of Heisenberg picture it is easy to check that Eq. (1.59) is equivalent to

$$\mathscr{C}(A) = \mathrm{Tr}_{\mathcal{E}_{\rm out}}[VAV^\dagger] \quad \forall\, A \in \mathcal{T}(\mathcal{H}_{\rm in}). \tag{1.64}$$

In fact, from Eq. (1.52) we have

$$\begin{aligned}\mathrm{Tr}[B\mathscr{C}(A)] &= \mathrm{Tr}[\mathscr{C}^\top(B)A] = \\ &= \mathrm{Tr}[V^\dagger(B \otimes \mathbb{1}_{\mathcal{E}_{\rm out}})VA] = \\ &= \mathrm{Tr}[(B \otimes \mathbb{1}_{\mathcal{E}_{\rm out}})VAV^\dagger] = \\ &= \mathrm{Tr}[B\mathrm{Tr}_{\mathcal{E}_{\rm out}}[VAV^\dagger]] \quad \forall\, B \in \mathcal{T}(\mathcal{H}_{\rm out}),\end{aligned} \tag{1.65}$$

for all $A \in \mathcal{T}(\mathcal{H}_{\rm in})$.

Now, we realize that we can always enlarge $\mathcal{E}_{\rm out}$ at will: this just sums up to adding zero columns to the isometry V. Thus, it is not a loss of generality to suppose that $d_{\mathcal{E}_{\rm out}}$ is some multiple of $d_{\mathcal{H}_{\rm in}}$. Then, we can consider another Hilbert space $\mathcal{E}_{\rm in}$, with dimension

$$d_{\mathcal{E}_{\rm in}} \doteq \frac{d_{\mathcal{H}_{\rm out}} \cdot d_{\mathcal{E}_{\rm out}}}{d_{\mathcal{H}_{\rm in}}}, \tag{1.66}$$

and let $|\psi\rangle$ be any pure state ($\langle\psi|\psi\rangle = 1$) in $\mathcal{E}_{\rm in}$. Finally, one can always find a linear operator U_ψ of $\mathcal{H}_{\rm in} \otimes \mathcal{E}_{\rm in}$ into $\mathcal{H}_{\rm out} \otimes \mathcal{E}_{\rm out}$ such that

$$U_\psi(\mathbb{1}_{\mathcal{H}_{\rm in}} \otimes |\psi\rangle_{\mathcal{E}_{\rm in}}) = V. \tag{1.67}$$

This yields Eq. (1.63), with

$$\begin{aligned}\mathbb{1}_{\mathcal{H}_{\rm in}} &= V^\dagger V = \\ &= (\mathbb{1}_{\mathcal{H}_{\rm in}} \otimes {}_{\mathcal{E}_{\rm in}}\langle\psi|)U_\psi^\dagger U_\psi(\mathbb{1}_{\mathcal{H}_{\rm in}} \otimes |\psi\rangle_{\mathcal{E}_{\rm in}}),\end{aligned} \tag{1.68}$$

i.e. $U_\psi^\dagger U_\psi = \mathbb{1}_{\mathcal{H}_{\text{in}}} \otimes \mathbb{1}_{\mathcal{E}_{\text{in}}}$: this proves that U_ψ is a unitary transformation. ■

Remark 1.12 Consider a CP&TP map $\mathscr{C}$ of $\mathcal{T}(\mathcal{H}_{\text{in}})$ into $\mathcal{T}(\mathcal{H}_{\text{out}})$, and let $\{M_x \mid x \in X\}$ be a Kraus decomposition for it: in general, a Kraus decomposition of a CP&TP map may consist of any (finite) number $|X|$ of linear transformations M_x. However (see Remark 1.5), the choice of treating with canonical decompositions, i.e. with linearly independent sets $\{M_x\}$, is not a restrictive one. This places an upper bound on the number of Kraus operators, namely $|X| \leq d_{\mathcal{H}_{\text{in}}} \cdot d_{\mathcal{H}_{\text{out}}}$.

Now, from the proof of Theorem 1.7, we know that the minimal dimension of $\mathcal{E}_{\text{out}}$ is $|X|$: in fact, if $d_{\mathcal{E}_{\text{out}}} < |X|$, we would have no natural way to define the isometry $V : \mathcal{H}_{\text{in}} \to \mathcal{H}_{\text{out}} \otimes \mathcal{E}_{\text{out}}$ satisfying Eq. (1.59). Then we see that, once fixed $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$, if we want $\mathcal{E}_{\text{out}}$ to fit all possible CP&TP maps $\mathscr{C}$, we must suppose $|X| = d_{\mathcal{H}_{\text{in}}} \cdot d_{\mathcal{H}_{\text{out}}}$ (worst case choice), so that the minimal dimension for $\mathcal{E}_{\text{out}}$ is $d_{\mathcal{H}_{\text{in}}} \cdot d_{\mathcal{H}_{\text{out}}}$.

Furthermore, in order to write Eq. (1.63), we have supposed that $d_{\mathcal{E}_{\text{out}}}$ is some multiple of $d_{\mathcal{H}_{\text{in}}}$, so that we may retain our optimal choice $d_{\mathcal{E}_{\text{out}}} = d_{\mathcal{H}_{\text{in}}} \cdot d_{\mathcal{H}_{\text{out}}}$ and, from Eq. (1.66), we obtain that the minimal dimension for $\mathcal{E}_{\text{in}}$ is $d^2_{\mathcal{H}_{\text{out}}}$. ▲

Now, Quantum Maps were introduced as a mathematical tool to describe state transformations of quantum systems. By Axioms 1.1, 1.2 and 1.4, *de facto* we have required all Quantum Maps to be CP&TP maps; furthermore, Corollary 1.8 clearly tells us that, conversely, all CP&TP maps may be regarded as state evolutions of open quantum system, namely all CP&TP maps deserve the adjective 'Quantum Map'. Thus, we have proved that the set of Quantum Maps (once one assumes Axioms 1.1, 1.2 and 1.4) concides with that of CP&TP maps, i.e. with the set of state evolutions of open systems.

In literature, CP&TP maps are known as Quantum Channels, so we give the following definition.

Definition 1.7 (Quantum Channel) *A Quantum Channel between Hilbert spaces $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$ is any* CP&TP *map $\mathscr{C} : \mathcal{T}(\mathcal{H}_{\text{in}}) \to \mathcal{T}(\mathcal{H}_{\text{out}})$, so that we shall write*

$$\mathrm{QC}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}) \doteq \mathrm{CP}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}) \cap \mathrm{TP}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}) \tag{1.69}$$

with an obvious meaning of symbols.

Indeed, in the framework of Quantum Information, when some information is being transmitted its physical storing device (say, a qubit) is assumed to be an open system, in order to take into account noise effects coming from environment or third parties.

Remark 1.13 Notice that Quantum Channels generalize Quantum States. Indeed, consider a linear map $\mathscr{C}$ between $\mathbb{C}$ and $\mathcal{H}$. Then, its Choi operator is given by $R_{\mathscr{C}} = \mathscr{C}(1) \in \mathcal{B}(\mathcal{H})$, so that $\mathscr{C}$ is a Quantum Channel if and only if

$$\begin{cases} \mathscr{C}(1) \geq 0, \\ \mathrm{Tr}[\mathscr{C}(1)] = 1, \end{cases} \tag{1.70}$$

namely if and only if $\mathscr{C}(1) \in \mathrm{S}(\mathcal{H})$. Thus, we see that the set of states $\mathrm{S}(\mathcal{H})$ may be seen as the set of Quantum Channels taking $\mathbb{C}$ in $\mathcal{H}$:

$$\mathrm{S}(\mathcal{H}) \cong \mathrm{QC}(\mathbb{C}, \mathcal{H}). \tag{1.71}$$

▲

1.3.2 On the Convex Set of Quantum Channels

As we have seen above, both sets of CP and of TP maps are convex, so that their intersection (i.e. the set of Quantum Channels) is convex as well. In fact, using Choi isomorphism we may write

$$\begin{aligned} \mathrm{QC}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}) &= \mathrm{CP}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}) \cap \mathrm{TP}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}) \cong \\ &\cong \Omega(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}) \cap \mathrm{N}_{\mathbb{1}_{\mathrm{in}}}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}). \end{aligned} \tag{1.72}$$

As a result, all convex combinations of Quantum Channels are Quantum Channels as well and, conversely, all Quantum Channels may be decomposed into some (proper or trivial) convex combination of Quantum Channels.

The physical interpretation of convex combinations of Quantum Channels is straightforward: indeed, as randomization of input states is obtained by considering ensembles of input density operators $\{(\rho_i, p_i) \mid i \in I\}$, randomization of output states is obtained considering ensembles of Quantum Channels $\{(\mathscr{C}_j, p'_j) \mid j \in J\}$, such that

$$\mathscr{C}(\rho) = \sum_{j \in J} p'_j \mathscr{C}_j(\rho). \tag{1.73}$$

This shows that convex combinations of Quantum Channels may be regarded as their randomization: extremal elements in the set of Quantum Channels, thus, admit no description in terms of randomized Quantum Channels. Such extremal Quantum Channels are characterized by a Theorem once again due to Choi [2]:

Theorem 1.9 (Extremal Quantum Channels) *Let $\mathscr{C}$ be a Quantum Channel of $\mathcal{T}(\mathcal{H}_{\text{in}})$ into $\mathcal{T}(\mathcal{H}_{\text{out}})$, and let $\{M_x \mid x \in X\}$ be one of its Kraus canonical decompositions. Then, $\mathscr{C}$ is extremal in* $\mathrm{QC}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}})$ *if and only if $\{M_x^\dagger M_y \mid x, y \in X\}$ is a linearly independent subset of $\mathcal{T}(\mathcal{H}_{\text{in}})$.*

Proof Let us work in the Heisenberg picture: then, we have the CP, unital map $\mathscr{C}^\top$ of $\mathcal{B}(\mathcal{H}_{\text{out}})$ into $\mathcal{B}(\mathcal{H}_{\text{in}})$, with Kraus decomposition $\{M_x^\dagger \mid x \in X\}$. Clearly, $\mathscr{C}^\top$ is extremal in the set of CP, unital maps if and only if $\mathscr{C}$ is extremal in the set of CP, TP maps.

So, let us assume that $\mathscr{C}^\top$ is extremal: then, we want to prove that the only way to write

$$\sum_{x,y \in X} \lambda_{x,y} M_x^\dagger M_y = 0 \tag{1.74}$$

is the trivial one, namely $\Lambda = 0$ – where $\Lambda = (\lambda_{x,y})_{x,y}$. Once fixed a matrix Λ such that Eq. (1.74) holds, we note that taking the Hermitian conjugate of it yields

$$\sum_{x,y \in X} \lambda_{y,x}^* M_x^\dagger M_y = 0, \tag{1.75}$$

so that, taking the sum and the difference of the two, we have

$$\sum_{x,y \in X} (\lambda_{x,y} \pm \lambda_{y,x}^*) M_x^\dagger M_y = 0. \tag{1.76}$$

Thus, we must prove that $\Lambda_\pm = 0$, with $\Lambda_\pm = \Lambda \pm \Lambda^\dagger$: this is equivalent to prove that $\Lambda = 0$ with $\Lambda = \Lambda^\dagger$. By a scalar multiplication, we may further assume $-\mathbb{1} \leq \Lambda \leq \mathbb{1}$. Now, let us define maps $\mathscr{C}_\pm^\top$ as

$$\mathscr{C}_\pm^\top(B) = \sum_{x,y \in X} (\mathbb{1} \pm \Lambda)_{x,y} M_x^\dagger B M_y \quad \forall\, B \in \mathcal{B}(\mathcal{H}_{\text{out}}); \tag{1.77}$$

then,

$$\begin{aligned} \mathscr{C}_\pm^\top(\mathbb{1}_{\mathcal{H}_{\text{out}}}) &= \sum_{x,y \in X} M_x^\dagger M_x \pm \sum_{x,y \in X} \lambda_{x,y} M_x^\dagger M_x = \\ &= \mathbb{1}_{\mathcal{H}_{\text{in}}}, \end{aligned} \tag{1.78}$$

i.e. $\mathscr{C}_\pm^\top$ are unital maps. Furthermore, they are CP as well: in fact, let

$$\begin{cases} \mathbb{1} \pm \Lambda = \Gamma_\pm^\dagger \Gamma_\pm, \\ N_{\pm,z} \doteq \displaystyle\sum_{x \in X} \gamma_{z,x}^\pm M_x, \quad z \in Z, \end{cases} \tag{1.79}$$

where $\Gamma_\pm = (\gamma^\pm_{z,x})_{z,x}$ is a $|Z| \times |X|$ matrix. Then

$$\begin{aligned} \mathscr{C}_\pm^\top(B) &= \sum_{x,y\in X} (\Gamma_\pm^\dagger \Gamma_\pm)_{x,y} M_x^\dagger B M_y = \\ &= \sum_{z\in Z} \sum_{x\in X} \gamma^{\pm *}_{z,x} M_x^\dagger \; B \sum_{y\in X} \gamma^\pm_{z,y} M_y = \\ &= \sum_{z\in Z} N_{\pm,z}^\dagger B N_{\pm,z} \quad \forall\, B \in \mathcal{B}(\mathcal{H}_{\text{out}}). \end{aligned} \tag{1.80}$$

Since $\mathscr{C}^\top = \frac{1}{2}(\mathscr{C}_+^\top + \mathscr{C}_-^\top)$, with $\mathscr{C}^\top$ extremal, we obtain $\mathscr{C}^\top = \mathscr{C}_+^\top = \mathscr{C}_-^\top$: this means that $\{M_x \mid x \in X\}$ and $\{N_{\pm,z} \mid z \in Z\}$ are Kraus decompositions of the same map, so that, thanks to Theorem 1.4, matrices $\Gamma_\pm$ are isometries. Thus, recalling the first of Eqq. (1.79) yields $\mathbb{1} \pm \Lambda = \mathbb{1}$, i.e. $\Lambda = 0$.

Now, let us assume that $\{M_x^\dagger M_y \mid x, y \in X\}$ is a linearly independent set: then, $\{M_x \mid x \in X\}$ is a linearly independent set too. Furthermore, let $\mathscr{C}^\top = \frac{1}{2}(\mathscr{C}_1^\top + \mathscr{C}_2^\top)$ for some extremal maps $\mathscr{C}_1^\top$, $\mathscr{C}_2^\top$ with Kraus decompositions respectively given by $\{N^{(1)}_{z_1} \mid z_1 \in Z_1\}$ and $\{N^{(2)}_{z_2} \mid z_2 \in Z_2\}$, and let $\Gamma^{(1)}$ and $\Gamma^{(2)}$ be, respectively, $|Z_1| \times |X|$ and $|Z_2| \times |X|$ matrices such that

$$N^{(i)}_{z_i} = \sum_{x\in X} \Gamma^{(i)}_{z_i,x} M_x \quad \forall\, z_i \in Z_i, \quad i \in \{1,2\}. \tag{1.81}$$

Then, unital conditions of maps $\mathscr{C}_i^\top$ read

$$\begin{aligned} \mathbb{1}_{\mathcal{H}_{\text{in}}} &= \sum_{z_i\in Z_i} N^{(i)\dagger}_{z_i} N^{(i)}_{z_i} = \\ &= \sum_{z_i\in Z_i} \sum_{x\in X} \Gamma^{(i)*}_{z_i,x} M_x^\dagger \sum_{y\in X} \Gamma^{(i)}_{z_i,y} M_y = \\ &= \sum_{x,y\in X} (\Gamma^{(i)\dagger}\Gamma^{(i)})_{x,y} M_x^\dagger M_y, \quad i \in \{1,2\}, \end{aligned} \tag{1.82}$$

whilst unital condition of $\mathscr{C}^\top$ reads

$$\sum_{x\in X} M_x^\dagger M_x = \mathbb{1}_{\mathcal{H}_{\text{in}}}. \tag{1.83}$$

Comparing the two last Eqq. yields, thanks to the hypothesis of linearly independence of $\{M_x^\dagger M_y \mid x, y \in X\}$,

$$(\Gamma^{(i)\dagger}\Gamma^{(i)})_{x,y} = \delta_{x,y}, \quad i \in \{1,2\}, \tag{1.84}$$

i.e. $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are isometric matrices. Then, by Theorem 1.4, we have that $\mathscr{C}^\top = \mathscr{C}_1^\top = \mathscr{C}_2^\top$, so that $\mathscr{C}^\top$ is extremal as well. ■

Remark 1.14 Theorem 1.9 lets us place a stricter bound on the number $|X|$ of Kraus transformations for extremal Quantum Channels: indeed, if $\{M_x^\dagger M_y\}$ is a linearly independent set on $\mathcal{T}(\mathcal{H}_{\text{in}})$, then the number of its element ($|X|^2$) must not exceed the dimension of $\mathcal{B}(\mathcal{H}_{\text{in}})$ (d_{in}^2), so that $|X| \leq d_{\text{in}}$ is a necessary condition for $\mathscr{C}$ to be extremal.

Recalling Remark 1.12, we see that all extremal Quantum Channels may be seen as unitary transformations of $\mathcal{H}_{\text{in}} \otimes \mathcal{E}_{\text{in}}$ into $\mathcal{H}_{\text{out}} \otimes \mathcal{E}_{\text{out}}$ with $\mathcal{E}_{\text{out}} \cong \mathcal{H}_{\text{in}}$ and $\mathcal{E}_{\text{in}} \cong \mathcal{H}_{\text{out}}$. ▲

1.3.3 Notes on Trace-Decreasing Maps

As we pointed out before, it is just a matter of convenience whether to use Choi operators or Kraus decompositions in order to characterize CP maps, so the same holds for the characterization of Quantum Channels. Nevertheless, as anticipated in Remark 1.6, Kraus decompositions carry an explicit physical interpretation of state evolution.

Indeed, let $\mathscr{C}$ be a Quantum Channel of $\mathcal{T}(\mathcal{H}_{\text{in}})$ into $\mathcal{T}(\mathcal{H}_{\text{out}})$ with Kraus decomposition $\{M_x \mid x \in X\}$, and let us rewrite its action (1.31) on a state $\rho \in \mathrm{S}(\mathcal{H}_{\text{in}})$ as

$$\mathscr{C}(\rho) = \sum_{x \in X} p_x(\rho) \frac{\mathscr{E}_x(\rho)}{p_x(\rho)}, \tag{1.85}$$

where we have put

$$\begin{cases} \mathscr{E}_x(\rho) \doteq M_x \rho M_x^\dagger, \\ p_x(\rho) \doteq \mathrm{Tr}[\mathscr{E}_x(\rho)]. \end{cases} \tag{1.86}$$

Then, of course $[p_x(\rho)]^{-1}\mathscr{E}_x(\rho)$ satisfies conditions (1.2) for all $x \in X$ and for all $\rho \in \mathrm{S}(\mathcal{H}_{\text{in}})$, namely it is a proper density operator in $\mathrm{S}(\mathcal{H}_{\text{out}})$; furthermore, since $\mathscr{E}_x(\rho_{\text{in}})$ is positive, then $p_x(\rho_{\text{in}}) \geq 0$, and TP condition (1.48) guarantees that

$$\begin{aligned} \sum_{x \in X} p_x(\rho) &= \mathrm{Tr}\left[\sum_{x \in X} M_x^\dagger M_x \rho\right] \\ &= \mathrm{Tr}[\rho] = \\ &= 1, \end{aligned} \tag{1.87}$$

i.e. $\{p_x(\rho) \mid x \in X\}$ may be interpreted as probabilities for all $\rho \in \mathrm{S}(\mathcal{H}_{\text{in}})$. Thus, Kraus decomposition tells us that output of Quantum Channels may be seen as the randomization, with certain probabilities $\{p_x(\rho) \mid x \in X\}$ depending on the initial state, of the transformation

$$\rho \mapsto \frac{\mathscr{E}_x(\rho)}{p_x(\rho)}. \tag{1.88}$$

In the framework of quantum measurement theory, each of the above *non-linear* transformations is recognized as a *state reduction* of the quantum system: in the case of absence of measurement, then, we see that Quantum Channels may be regarded as randomizations of state reductions.

Remark 1.15 Though decomposition (1.85) may be seen as a convex combination of elements $p_x(\rho)^{-1}\mathscr{E}_x(\rho)$, such elements are *not* part of $\mathrm{QC}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}})$ anymore (more on this later). Thus, Eq. (1.85) has nothing to do with Eq. (1.73), representing the randomization of Quantum Channels. In fact, whilst extremal Quantum Channels do not admit any proper decomposition (1.73), they still admit decomposition (1.85), with the only constraint that $\{M_x^\dagger M_y \mid x, y \in X\}$ is a linearly independent set (see Theorem 1.9). ▲

Now, if we regard the environment as (containing) a measurement apparatus, we are allowed to consider the case in which some information is gained on the specific transformation the system and the apparatus have jointly undergone. For instance, by looking at the pointer we might be able to say that, of all the possible transformations (each corresponding to an index $x \in X$), the $\overline{x}$-th has taken place. In this case, we are allowed to select the sub-ensemble described by $\mathrm{Tr}[\mathscr{E}_{\overline{x}}(\rho)]^{-1}\mathscr{E}_{\overline{x}}(\rho)$ as the output state, instead of the randomization of output state of all possible transformations.

Then, we have heuristically shown that measurements situations – where one gains some information on the interaction between the system and some apparatus, and is thus provided a rule to select sub-ensembles of states – may be described by non-linear transformations (1.88), where the non-linearity is due to the renormalization which is needed when one selects sub-ensembles.

Maps $\mathscr{E}_x$ are evidently CP: on the other hand, it is straightforward to realize that they are not TP in general (they are TP only in the trivial case $|X| = 1$). In literature, they are known as Trace-Decreasing (TD) maps[6]. Clearly, CP&TD maps $\mathscr{E}_x$ do not comply with Axiom 1.1, as they do not map Quantum States into Quantum States. Still, insted of maps $\mathscr{E}_x(\rho)$, one may consider normalized maps $p_x(\rho)^{-1}\mathscr{E}_x(\rho)$, which preserve states: unfortunately, such maps contradict Axiom 1.2, as they are no more convex-linear on the set of states.

Thus, we stress the fact that formally, according to our previous axiomatization, CP&TD maps do not meet our requirements for Quantum Maps. However, let us stress the fact that CP&TD maps do describe physical transformations of quantum states and, in particular, they describe measurement-induced evolutions.

[6]Though it would be more proper to say that they *do not increase* the trace.

Chapter 2

Quantum Supermaps

In the present Chapter, the notion of Quantum Supermaps is introduced as a mathematical tool for the study of Quantum Maps' transformations. Notice that the structure of this Chapter closely recalls that of Chapter 1: indeed, the axiomatization of Quantum Supermaps being presented in Section 2.1 is carried out in strict analogy with that of Quantum Maps (see Section 1.1), and the properties of Quantum Supermaps are investigated in Section 2.2 with a constant regard to analogous features of Quantum Maps (see Section 1.2). Furthermore, Section 2.4 concludes this Chapter with the important study of the relation between the mathematical formalism of Quantum Supermaps and their physical implementation (as for the case of Quantum Maps, see Section 1.3). An exception to the parallel structures of Chapters 1 and 2 is represented by Section 2.3, where covariant supermaps are introduced mainly as a preparatory study for 1-to-2 Unitary Cloning Supermaps, that are presented in Chapter 3.

2.1 Transformations of Quantum Maps

As we have seen in Chapter 1, Quantum Maps are introduced as a mathematical formalism to describe state transformations of quantum-mechanical systems. However, the striking similarity between the structure of the resulting set of Quantum Maps and that of Quantum States suggests that, regarding Quantum Maps as being "states of state transformations" (*de facto*, as super-states), we could adapt most of the notions that were introduced in Chapter 1 to the study of such super-states.

In the following, in perfect analogy with Chapter 1, we will use the generic term 'Quantum Supermap' to describe mathematical super-maps on the set of Quantum Maps (i.e. of super-states) describing all of their physical transformations. Of course, the notion of physical transformations of Quantum Maps may look, at first sight, at least puzzling: in fact, in the case of Quantum Maps we did expect the most general state transformation to be a state evolution of an open system, whilst here it is more difficult to have a physical guess *a priori* on the most general transformation of Quantum Channels. Nevertheless, the hypothesis that each and every Quantum Map should represent some state evolution of open systems was never exploited in Chapter 1: similarly, ... In the end, we will show ...

2.1.1 Quantum Channel-Preserving Supermaps

For the sake of clarity, as in Chapter 1 we implicitly distinguished between *operators* representing states and *maps* representing super-operators, here we will use the term 'supermap' to denote super-super-operators, namely maps $\mathbb{S}$ with domain in $\mathcal{L}(\mathcal{T}(\mathcal{H}_{\mathrm{in}}), \mathcal{T}(\mathcal{H}_{\mathrm{out}}))$ and range in $\mathcal{L}(\mathcal{T}(\mathcal{H}_{\mathrm{in}'}), \mathcal{T}(\mathcal{H}_{\mathrm{out}'}))$, for some Hilbert spaces $\mathcal{H}_{\mathrm{in}}$, $\mathcal{H}_{\mathrm{out}}$, $\mathcal{H}_{\mathrm{in}'}$ and $\mathcal{H}_{\mathrm{out}'}$. For brevity, given such a supermap we will also say that $\mathcal{H}_{\mathrm{in}}$ and $\mathcal{H}_{\mathrm{out}}$ are its input spaces, whilst $\mathcal{H}_{\mathrm{in}'}$ and $\mathcal{H}_{\mathrm{out}'}$ are its output ones. Furthermore, for any generic supermap $\mathbb{S}$, when Hilbert spaces are not explicitly specified, we will denote its input (output) spaces with $\mathcal{H}_{\mathrm{in}}$, $\mathcal{H}_{\mathrm{out}}$ ($\mathcal{H}_{\mathrm{in}'}$, $\mathcal{H}_{\mathrm{out}'}$).

Since Quantum Supermaps must represent physical transformations of Quantum Maps, the least we may ask is

Axiom 2.1 *All Quantum Supermaps must preserve Quantum Maps.*

Of course, Axiom 2.1 is perfectly analogous to Axiom 1.1 on page 4: then, we may wonder whether it is meaningfull to rephrase also Axiom 1.2 for the case of supermaps. Of course, the answer is a positive one: indeed, if the input Quantum Channel $\mathscr{C}$ describes a statistical ensemble (i.e. a randomization) $\{(\mathscr{C}_i, p_i) \mid i \in I\}$ of Quantum Channels, then the output Quantum Channel $\mathbb{S}(\mathscr{C})$ must describe the ensemble $\{(\mathbb{S}(\mathscr{C}_i), p_i) \mid i \in I\}$. So, we state the following

Axiom 2.2 *All Quantum Supermaps must be convex-linear on the set of Quantum Maps.*

Thus, all Quantum Supermaps must be convex-linear supermaps of $\mathrm{QC}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}})$ into some subset of $\mathrm{QC}(\mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'})$: moreover, since all of such convex-linear supermaps admit a linear extension to the whole $\mathcal{L}(\mathcal{T}(\mathcal{H}_{\mathrm{in}}), \mathcal{T}(\mathcal{H}_{\mathrm{out}}))$[1], then it is not a loss of generality to consider only linear

[1]As in the case of Quantum Maps: see Footnote 2 on page 5.

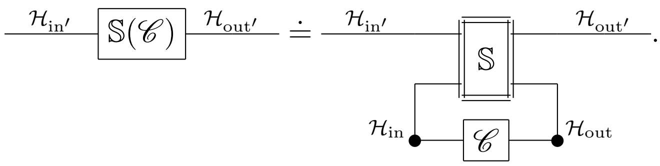
supermaps of $\mathcal{L}(\mathcal{T}(\mathcal{H}_{\text{in}}), \mathcal{T}(\mathcal{H}_{\text{out}}))$ into $\mathcal{L}(\mathcal{T}(\mathcal{H}_{\text{in}'}), \mathcal{T}(\mathcal{H}_{\text{out}'}))$, as long as they are Quantum-Channel Preserving:

Definition 2.1 (QCP **Supermap)** *Let $\mathbb{S}$ be a linear supermap. We will say that it is* Quantum Channel-Preserving, *or* QCP, *when*

$$\mathbb{S}\big(\mathrm{QC}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}})\big) \subseteq \mathrm{QC}(\mathcal{H}_{\text{in}'}, \mathcal{H}_{\text{out}'}), \tag{2.1}$$

and we will denote with $\mathrm{QCP}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}; \mathcal{H}_{\text{in}'}, \mathcal{H}_{\text{out}'})$ *the set of* QCP *supermaps taking $\mathcal{L}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}})$ into $\mathcal{L}(\mathcal{H}_{\text{in}'}, \mathcal{H}_{\text{out}'})$.*

In the following, when a QCP supermap $\mathbb{S}$ is fed with a Quantum Channel $\mathscr{C}$, the output Quantum Channel $\mathbb{S}(\mathscr{C})$ will be diagramatically represented by



(2.2)

Remark 2.1 Note that the right-hand side of Eq. (2.2) must be intended as an abstract diagram: in fact, we cannot say *a priori* whether all output Quantum Channels $\mathbb{S}(\mathscr{C})$ of all QCP supermaps $\mathbb{S}$ can be equivalently obtained using a quantum circuit of which $\mathscr{C}$ is a composing gate, as the diagram suggests. However, in Section 2.4 we will prove that the answer to such a question is a positive one, so that this particular choice of diagramatical representation will be justified. ▲

2.1.2 On CP- and TP-Preserving Supermaps

Recall Subsection 1.1.3: there, P and TP maps were introduced for the simple reason that requiring a map to be P&TP was a sufficient condition for it to be State-Preserving. Later, it was proved that the condition was necessary as well (see Lemma 1.1), so that the set of SP maps coincided with that of P&TP ones.

Here we do something similar: indeed, since the set of Quantum Channels is obtained as the intersection between the two sets of CP and TP maps, then a sufficient condition for a supermap $\mathbb{S}$ to preserve Quantum Channels is given by the two independent preservations of CP and TP maps: this leads to the two following definitions.

Definition 2.2 (CP^2 Supermap) *Let $\mathbb{S}$ be a supermap: we will say that $\mathbb{S}$ is* Complete-Positivity Preserving, *or* CP^2, *when it preserves the Complete Positivity of maps, namely when*

$$\mathbb{S}\big(\mathrm{CP}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}})\big) \subseteq \mathrm{CP}(\mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'}). \tag{2.3}$$

We will denote the set of CP^2 *supermaps with* $\mathrm{CP}^2(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}; \mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'})$.

Definition 2.3 (TP^2 Supermap) *Let $\mathbb{S}$ be a supermap: we will say that $\mathbb{S}$ is* Trace-Preservation Preserving, *or* TP^2, *when it preserves* TP *maps, namely when*

$$\mathbb{S}\big(\mathrm{TP}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}})\big) \subseteq \mathrm{TP}(\mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'}). \tag{2.4}$$

We will denote the set of TP^2 *supermaps with* $\mathrm{TP}^2(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}; \mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'})$.

However, our parallelism between Quantum Maps and Quantum Supermaps reaches a stop here, due to the fact that, whilst in Lemma 1.1 on page 7 it was rather easy to show that the P&TP conditions were also necessary for a map to be SP, apparently now we have no way to prove the analogous result for supermaps, namely we cannot prove that CP^2&TP^2 conditions are also necessary for a supermap to be QCP. In fact, though we shall show that all QCP supermaps are TP^2 as well (see Lemma 2.9 on page 40), in the present treating we shall not give a proof of the fact that all QCP supermaps are CP^2.

On the other hand, recalling Subsection 1.3.3, the fact that all Quantum Supermaps must preserve the CP character of Trace-Decreasing maps is a reasonable requirement: indeed, if this were not true, every single state reduction in the form (1.88) would be mapped into an unphysical transformation. Thus, we state the following

Axiom 2.3 *All Quantum Supermaps must inject* CP&TD *maps into* CP&TD *maps.*

Then, since TD maps have no normalization condition (*modulo* a scaling factor) we state the following

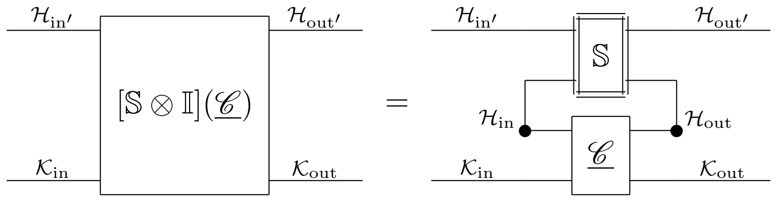
Proposition 2.4 *All Quantum Supermaps are* CP^2&TP^2.

which supersedes Axioms 2.1 and 2.2.

Clearly, Proposition 2.4 is analogous to Proposition 1.3, so that our parallelism between Quantum Maps and Quantum Supermaps is fully restored.

2.1.3 On Completely CP-Preserving Supermaps

Now, let us consider a Quantum Supermap $\mathbb{S}$ acting on maps of $\mathcal{T}(\mathcal{H}_{\text{in}})$ into $\mathcal{T}(\mathcal{H}_{\text{out}})$, and returning maps of $\mathcal{T}(\mathcal{H}_{\text{in}'})$ into $\mathcal{T}(\mathcal{H}_{\text{out}'})$. Then, if $\underline{\mathscr{C}}$ is some Quantum Channel of $\mathcal{T}(\mathcal{H}_{\text{in}} \otimes \mathcal{K}_{\text{in}})$ into $\mathcal{T}(\mathcal{H}_{\text{out}} \otimes \mathcal{K}_{\text{out}})$, we may consider its transformation under the action of the supermap $\mathbb{S} \otimes \mathbb{I}$, where $\mathbb{I}$ is the identity operator on the space of maps of $\mathcal{T}(\mathcal{K}_{\text{in}})$ into $\mathcal{T}(\mathcal{K}_{\text{out}})$. The resulting map is depicted in the following diagramatic equation:



. (2.5)

Of course, then, it is natural to require $[\mathbb{S} \otimes \mathbb{I}](\underline{\mathscr{C}})$ to be a Quantum Channel for all Quantum Supermaps $\mathbb{S}$ and for all Quantum Channels $\mathscr{C}$: thus we state the following

Axiom 2.5 *All Quantum Supermaps $\mathbb{S}$ must be such that all their trivial extensions $\mathbb{S} \otimes \mathbb{I}$ are Quantum Supermaps as well.*

Evidently, Axiom 2.5 is perfectly analogous to Axiom 1.4.

Here we find another striking similarity between Quantum Maps and Quantum Supermaps: indeed, just as the trivial extension $\mathscr{C} \otimes \mathscr{I}$ of P&TP maps $\mathscr{C}$ would still be TP, but would not be necessarily P anymore, the extension $\mathbb{S} \otimes \mathbb{I}$ of a CP^2&TP^2 supermap $\mathbb{S}$ is still TP^2 (the proof will be given in Lemma 2.10 on page 41), but is not necessarily CP^2 anymore. Clearly, then, if we want a CP^2&TP^2 supermap $\mathbb{S}$ to comply with Axiom 2.5, we must require it to be Completely CP^2, as specified by

Definition 2.4 (C^2P^2 **Supermap)** *Let $\mathbb{S}$ be a* CP^2 *supermap: we will say that it is* Completely CP^2*, or* C^2P^2*, when all its trivial extensions are* CP^2 *as well, namely when*

$$\mathbb{S} \otimes \mathbb{I} \in \text{CP}^2(\mathcal{H}_{\text{in}} \otimes \mathcal{K}_{\text{in}}, \mathcal{H}_{\text{out}} \otimes \mathcal{K}_{\text{out}}; \mathcal{H}_{\text{in}'} \otimes \mathcal{K}_{\text{in}}, \mathcal{H}_{\text{out}'} \otimes \mathcal{K}_{\text{out}}) \tag{2.6}$$

for all $\mathcal{K}_{\text{in}}$, $\mathcal{K}_{\text{out}}$, where $\mathbb{I}$ is the identical supermap on $\mathcal{L}(\mathcal{T}(\mathcal{K}_{\text{in}}), \mathcal{T}(\mathcal{K}_{\text{out}}))$. Furthermore, we will denote the set of C^2P^2 *supermaps with* $\text{C}^2\text{P}^2(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}; \mathcal{H}_{\text{in}'}, \mathcal{H}_{\text{out}'})$.

2.2 Characterization of Quantum Supermaps

The main purpose of the present Section is to obtain a handy mathematical characterization of C^2P^2 and TP^2 supermaps, just as was done in Section 1.2 for CP and TP maps. Besides, a few results that were previously anticipated are here thoroughly proved.

2.2.1 Choi Isomorphism for Supermaps

In the same way as Choi isomorphism allowed us us treat super-operators (maps) as operators, it turns out that it allows us to treat supermaps as maps:

Definition 2.5 (Representing Map) *Let* $\mathbb{S}$ *be a linear supermap: we will say that its* representing map in the space of Choi operators *(or just* representing map*) is the map* $\mathscr{S}_{\mathbb{S}}$ *that takes Choi operators corresponding to the input maps into those corresponding to the output ones:*

$$\mathscr{S}_{\mathbb{S}} : \begin{cases} \mathcal{B}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'}), \\ R_{\mathscr{C}} \mapsto R_{\mathbb{S}(\mathscr{C})} \quad \forall\, \mathscr{C} \in \mathcal{L}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}). \end{cases} \tag{2.7}$$

Furthermore, representing maps let us introduce Choi operators of supermaps in a natural way:

Definition 2.6 (Choi Operator of a Supermap) *Let* $\mathbb{S}$ *be a linear supermap, and let* $\{\mathscr{E}_{i,j;k,l} \mid i,j = 1,\dots,d_{\mathrm{out}}, k,l = 1,\dots,d_{\mathrm{in}}\}$ *denote the basis for the space of linear maps of* $\mathcal{T}(\mathcal{H}_{\mathrm{in}})$ *into* $\mathcal{T}(\mathcal{H}_{\mathrm{out}})$ *defined by*

$$\mathscr{E}_{i,j;k,l}(A) \doteq \langle k|A|l\rangle |i\rangle\langle j| \quad \forall\, A \in \mathcal{T}(\mathcal{H}_{\mathrm{in}}). \tag{2.8}$$

Then, Choi Operator of the supermap $\mathbb{S}$ *is given by*

$$R_{\mathbb{S}} \doteq \sum_{i,j=1}^{d_{\mathrm{out}}} \sum_{k,l=1}^{d_{\mathrm{in}}} R_{\mathbb{S}(\mathscr{E}_{i,j;k,l})} \otimes R_{\mathscr{E}_{i,j;k,l}}, \tag{2.9}$$

where $R_{\mathscr{C}}$*, in the right-hand side, denotes the usual Choi operator corresponding to the map* $\mathscr{C}$*.*

Remark 2.2 Definition 2.6 is such that Choi operators of supermaps coincide with those of their representative maps. In fact, Eq. (2.9) may be expanded as follows:

$$\begin{aligned}
R_{\mathbb{S}} &= \sum_{i,j=1}^{d_{\text{out}}} \sum_{k,l=1}^{d_{\text{in}}} R_{\mathbb{S}(\mathscr{E}_{i,j;k,l})} \otimes R_{\mathscr{E}_{i,j;k,l}} \\
&= \sum_{i,j=1}^{d_{\text{out}}} \sum_{k,l=1}^{d_{\text{in}}} \mathscr{S}_{\mathbb{S}}(R_{\mathscr{E}_{i,j;k,l}}) \otimes R_{\mathscr{E}_{i,j;k,l}} = \\
&= \sum_{i,j=1}^{d_{\text{out}}} \sum_{k,l=1}^{d_{\text{in}}} \mathscr{S}_{\mathbb{S}}\Big(|i\rangle_{\text{out}}\langle j| \otimes |k\rangle_{\text{in}}\langle l|\Big) \otimes |i\rangle_{\text{out}}\langle j| \otimes |k\rangle_{\text{in}}\langle l| = \\
&= [\mathscr{S}_{\mathbb{S}} \otimes \mathscr{I}]\,(|\mathbb{1}_{\text{out,in}}\rangle\rangle\langle\langle \mathbb{1}_{\text{out,in}}|)\,,
\end{aligned} \tag{2.10}$$

which is just the definition of Choi operator corresponding to the representative map $\mathscr{S}_{\mathbb{S}}$. ▲

The inverse isomorphism between Choi operators and supermaps is given by the following

Lemma 2.1 (Choi Isomorphism for Supermaps) *Supermaps $\mathbb{S}$ are in $1:1$ correspondence with operators $R_{\mathbb{S}} \in \mathcal{B}(\mathcal{H}_{\text{out}'} \otimes \mathcal{H}_{\text{in}'} \otimes \mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$ such that, for all input maps $\mathscr{C} \in \mathcal{L}(\mathcal{T}(\mathcal{H}_{\text{in}}), \mathcal{T}(\mathcal{H}_{\text{out}}))$, the output map acts like*

$$[\mathbb{S}(\mathscr{C})](A') = \mathrm{Tr}_{\text{in}',\text{out},\text{in}}\left[\left(\mathbb{1}_{\text{out}'} \otimes A'^{\top} \otimes R_{\mathscr{C}}^{\top}\right) R_{\mathbb{S}}\right] \tag{2.11}$$

on operators $A' \in \mathcal{T}(\mathcal{H}_{\text{in}'})$.

Proof The action of the output map $\mathbb{S}(\mathscr{C})$, in terms of its Choi operator $R_{\mathbb{S}(\mathscr{C})}$, is given by the inverse isomorphism formula (1.30),

$$[\mathbb{S}(\mathscr{C})](A') = \mathrm{Tr}_{\text{in}'}\left[\left(\mathbb{1}_{\text{out}'} \otimes A'^{\top}\right) R_{\mathbb{S}(\mathscr{C})}\right] \quad \forall\, A' \in \mathcal{T}(\mathcal{H}_{\text{in}'}). \tag{2.12}$$

Similarly, the action of the representing map $\mathscr{S}_{\mathbb{S}}$, in terms of its Choi operator $R_{\mathscr{S}_{\mathbb{S}}}$, is given by

$$\mathscr{S}_{\mathbb{S}}(R) = \mathrm{Tr}_{\text{out,in}}\left[\left(\mathbb{1}_{\text{out}'} \otimes \mathbb{1}_{\text{in}'} \otimes R^{\top}\right) R_{\mathscr{S}_{\mathbb{S}}}\right] \quad \forall\, R \in \mathcal{B}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}), \tag{2.13}$$

or, equivalently, thanks to Choi isomorphism, by

$$\mathscr{S}_{\mathbb{S}}(R_{\mathscr{C}}) = \mathrm{Tr}_{\text{out,in}}\left[\left(\mathbb{1}_{\text{out}'} \otimes \mathbb{1}_{\text{in}'} \otimes R_{\mathscr{C}}^{\top}\right) R_{\mathscr{S}_{\mathbb{S}}}\right] \quad \forall\, \mathscr{C} \in \mathcal{L}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}), \tag{2.14}$$

Since, by Definition 2.5 of representing maps, $\mathscr{S}_{\mathbb{S}}(R_{\mathscr{C}}) = R_{\mathbb{S}(\mathscr{C})}$, the combination of the above Eqq. yields

$$[\mathbb{S}(\mathscr{C})](A) = \mathrm{Tr}_{\text{in}',\text{out},\text{in}}\left[\left(\mathbb{1}_{\text{out}'} \otimes A^{\top} \otimes R_{\mathscr{C}}^{\top}\right) R_{\mathscr{S}_{\mathbb{S}}}\right] \quad \begin{array}{l}\forall\, \mathscr{C} \in \mathcal{L}(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}), \\ \forall\, A \in \mathcal{T}(\mathcal{H}_{\text{in}'}),\end{array} \tag{2.15}$$

so that, identifying $R_{\mathbb{S}}$ with the Choi operator of the representing map, $R_{\mathscr{S}_{\mathbb{S}}}$, we obtain the desired result. ■

Representing maps are generally easier to deal with than supermaps, just as Choi operators are easier to treat with than super-operators (maps): indeed, thanks to Theorem 1.3 and Lemma 1.5, QCP condition (2.1) may now be rewritten in the more convenient form

$$\mathscr{S}_{\mathbb{S}}\big([\Omega \cap \mathrm{N}_{\mathbb{1}_{\mathrm{in}}}](\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})\big) \subseteq [\Omega \cap \mathrm{N}_{\mathbb{1}_{\mathrm{in}'}}](\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'}). \tag{2.16}$$

Somehow, this recalls the definition of SP maps $\mathscr{C}$, that we rewrite here as

$$\mathscr{C}\big([\Omega \cap \mathrm{N}_1](\mathcal{H}_{\mathrm{in}})\big) \subseteq [\Omega \cap \mathrm{N}_1](\mathcal{H}_{\mathrm{out}}), \tag{2.17}$$

the main difference, of course, consisting of the different normalization for states and for Choi operators.

We now state a result that we will later need in order to characterize Quantum Supermaps.

Lemma 2.2 (Factorizable Supermaps) *Representing Maps of factorized Supermaps are factorized. In symbols,*

$$\mathscr{S}_{\mathbb{S}\otimes\mathbb{T}} = \mathscr{S}_{\mathbb{S}} \otimes \mathscr{S}_{\mathbb{T}}. \tag{2.18}$$

Proof This is straightforward since Choi isomorphism preserves factorizability. In fact, consider the factorized map $\mathscr{C} \otimes \mathscr{E}$, where $\mathscr{C}$ maps $\mathcal{T}(\mathcal{H}_{\mathrm{in}})$ into $\mathcal{T}(\mathcal{H}_{\mathrm{out}})$, and $\mathscr{E}$ maps $\mathcal{T}(\mathcal{K}_{\mathrm{in}})$ into $\mathcal{T}(\mathcal{K}_{\mathrm{out}})$. Then, from the very definition of Choi operators it is easy to check that

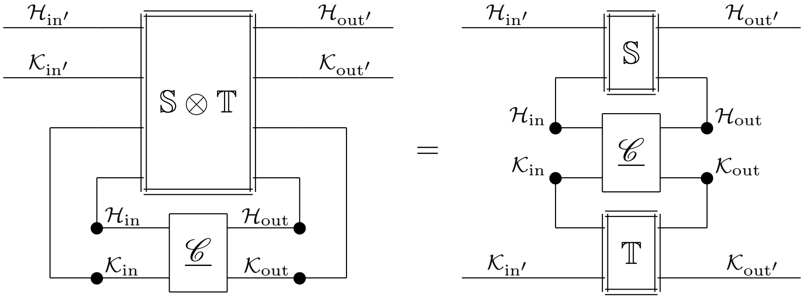
$$R_{\mathscr{C}\otimes\mathscr{E}} = R_{\mathscr{C}} \otimes R_{\mathscr{E}}. \tag{2.19}$$

Note that $R_{\mathscr{C}}$ acts on $\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}$, $R_{\mathscr{E}}$ on $\mathcal{K}_{\mathrm{out}} \otimes \mathcal{K}_{\mathrm{in}}$, whilst $R_{\mathscr{C}\otimes\mathscr{E}}$ on $\mathcal{H}_{\mathrm{out}} \otimes \mathcal{K}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}} \otimes \mathcal{K}_{\mathrm{in}}$, so that formally Hilbert spaces are ordered differently on the two sides of Eq. (2.19). Nevertheless, if $\mathbb{S}$ and $\mathbb{T}$ are allowed to act on $\mathscr{C}$ and $\mathscr{E}$, respectively, then we have

$$\begin{aligned} \mathscr{S}_{\mathbb{S}\otimes\mathbb{T}}\big(R_{\mathscr{C}\otimes\mathscr{E}}\big) &= R_{\mathbb{S}(\mathscr{C})\otimes\mathbb{T}(\mathscr{E})} = \\ &= R_{\mathbb{S}(\mathscr{C})} \otimes R_{\mathbb{T}(\mathscr{E})} = \\ &= \mathscr{S}_{\mathbb{S}}\big(R_{\mathscr{C}}\big) \otimes \mathscr{S}_{\mathbb{T}}\big(R_{\mathscr{E}}\big) = \\ &= [\mathscr{S}_{\mathbb{S}} \otimes \mathscr{S}_{\mathbb{T}}]\big(R_{\mathscr{C}} \otimes R_{\mathscr{E}}\big) = \\ &= [\mathscr{S}_{\mathbb{S}} \otimes \mathscr{S}_{\mathbb{T}}]\big(R_{\mathscr{C}\otimes\mathscr{E}}\big). \end{aligned} \tag{2.20}$$

Since non-factorized maps $\underline{\mathscr{C}}$ taking $\mathcal{T}(\mathcal{H}_{\mathrm{in}} \otimes \mathcal{K}_{\mathrm{in}})$ into $\mathcal{T}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{K}_{\mathrm{out}})$ may be written as linear combinations of factorized maps $\mathscr{C}_i \otimes \mathscr{E}_i$, then last Eq. proves our Lemma thanks to the linearity of representing maps $\mathscr{S}$. ■

Remark 2.3 Lemma 2.2 may be represented diagramatically as follows:



. (2.21)

This makes the ordering of Hilbert spaces in Eq. (2.18) more evident and easier to remember: indeed, whilst $\mathscr{S}_{\mathbb{S}\otimes\mathbb{T}}$ clearly maps $\mathcal{B}((\mathcal{H}_{\text{out}} \otimes \mathcal{K}_{\text{out}}) \otimes (\mathcal{H}_{\text{in}} \otimes \mathcal{K}_{\text{in}}))$ into its primed counterpart, $\mathscr{S}_{\mathbb{S}}$ acts only on the $\mathcal{H}$ part (taking $\mathcal{B}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$ into its primed counterpart), and $\mathscr{S}_{\mathbb{T}}$ on the $\mathcal{K}$ one (taking $\mathcal{B}(\mathcal{K}_{\text{out}} \otimes \mathcal{K}_{\text{in}})$ into its primed counterpart). ▲

2.2.2 Completely CP-Preserving Supermaps

In the present Subsection, we aim at obtaining a mathematical characterization of $\mathrm{C}^2\mathrm{P}^2$ supermaps, just as we did for CP maps in Subsection 1.2.2.

Though not all CP^2 supermaps are $\mathrm{C}^2\mathrm{P}^2$ as well, we need to characterize the former in order to study the latter. Furthermore, the study of CP^2 maps gives a beautiful example of the simplifications provided by representing supermaps: indeed, it is straightforward to realize that CP^2 condition (2.3) may be rephrased as

$$\mathscr{S}_{\mathbb{S}}\big(\Omega(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})\big) \subseteq \Omega(\mathcal{H}_{\text{out}'} \otimes \mathcal{H}_{\text{in}'}) \tag{2.22}$$

for any linear supermaps $\mathbb{S}$. Then, direct comparison of the last Eq. with Definition 1.2 of P maps yields

Lemma 2.3 (Characterization of CP^2 Supermaps) *Any supermap $\mathbb{S}$ is CP^2 if and only if its representing map $\mathscr{S}_{\mathbb{S}}$ is Positive: in symbols,*

$$\mathbb{S} \in \mathrm{CP}^2(\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}; \mathcal{H}_{\text{in}'}, \mathcal{H}_{\text{out}'}) \Leftrightarrow \mathscr{S}_{\mathbb{S}} \in \mathrm{P}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}'} \otimes \mathcal{H}_{\text{in}'}), \tag{2.23}$$

for all linear supermaps $\mathbb{S}$.

Now, even though the proof is no more so obvious, it turns out that also $\mathrm{C}^2\mathrm{P}^2$ supermaps admit a simple mathematical characterization in terms of their representing:

Theorem 2.4 (Characterization of $\mathrm{C^2P^2}$ Supermaps) *Any supermap $\mathbb{S}$ is $\mathrm{C^2P^2}$ if and only if its representing map $\mathscr{S}_{\mathbb{S}}$ is Completely Positive: in symbols,*

$$\mathbb{S} \in \mathrm{C^2P^2}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}; \mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'}) \Leftrightarrow \mathscr{S}_{\mathbb{S}} \in \mathrm{CP}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'}), \tag{2.24}$$

for all linear supermaps $\mathbb{S}$.

Proof Let us use Lemma 2.3 to rephrase $\mathrm{C^2P^2}$ condition (2.6) for $\mathbb{S}$ in terms of its representing map $\mathscr{S}_{\mathbb{S}}$: we obtain that $\mathbb{S}$ is $\mathrm{C^2P^2}$ if and only if

$$\mathscr{S}_{\mathbb{S}\otimes\mathbb{I}} \in \mathrm{P}\big((\mathcal{H}_{\mathrm{out}} \otimes \mathcal{K}_{\mathrm{out}}) \otimes (\mathcal{H}_{\mathrm{in}} \otimes \mathcal{K}_{\mathrm{in}}), (\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{K}_{\mathrm{out}}) \otimes (\mathcal{H}_{\mathrm{in}'} \otimes \mathcal{K}_{\mathrm{in}})\big) \tag{2.25}$$

for all $\mathcal{K}_{\mathrm{in}}$, $\mathcal{K}_{\mathrm{out}}$, where $\mathbb{I}$ is the identity on the space of maps $\mathcal{T}(\mathcal{K}_{\mathrm{in}}) \to \mathcal{T}(\mathcal{K}_{\mathrm{out}})$. Now, using Lemma 2.2, and the fact that

$$\mathscr{S}_{\mathbb{I}} = \mathscr{I}_{\mathcal{B}(\mathcal{K}_{\mathrm{out}}\otimes\mathcal{K}_{\mathrm{in}})}, \tag{2.26}$$

we obtain that $\mathrm{C^2P^2}$ condition (2.6) is equivalent to

$$\mathscr{S}_{\mathbb{S}} \otimes \mathscr{I} \in \mathrm{P}\big((\mathcal{H}_{\mathrm{out}} \otimes \mathcal{K}_{\mathrm{out}}) \otimes (\mathcal{H}_{\mathrm{in}} \otimes \mathcal{K}_{\mathrm{in}}), (\mathcal{K}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{out}'}) \otimes (\mathcal{K}_{\mathrm{in}} \otimes \mathcal{H}_{\mathrm{in}'})\big) \tag{2.27}$$

for all $\mathcal{K}_{\mathrm{in}}$, $\mathcal{K}_{\mathrm{out}}$, where $\mathscr{I}$ is the identity map on $\mathcal{B}(\mathcal{K}_{\mathrm{in}} \otimes \mathcal{K}_{\mathrm{out}})$: it is straightforward to check that this is exactly the Definition 1.4 of CP maps $\mathscr{S}_{\mathbb{S}}$. ■

Remark 2.4 In the following, we will call *Kraus decomposition of the* $\mathrm{C^2P^2}$ *supermap* $\mathbb{S}$ the usual Kraus decomposition $\{K_z \mid z \in Z\} \subset \mathcal{L}(\mathcal{B}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}), \mathcal{B}(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'}))$ of their representing CP map $\mathscr{S}_{\mathbb{S}}$. ▲

The following Corollary is trivial to derive, but it is also important:

Corollary 2.5 (to Theorem 2.4) *Any supermap $\mathbb{S}$ is $\mathrm{C^2P^2}$ if and only if its Choi operator $R_{\mathbb{S}}$ is positive.*

So, Theorem 2.4 has the important consequence that

$$\begin{aligned} \mathrm{C^2P^2}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}; \mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'}) &\cong \mathrm{CP}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'}) \cong \\ &\cong \Omega(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}), \end{aligned} \tag{2.28}$$

i.e. $\mathrm{C^2P^2}$ supermaps form a convex set.

2.2.3 TP-Preserving Supermaps

Exploiting the Choi isomorphism, and using Lemma 1.5, we may rewrite TP^2 condition (2.4) for $\mathbb{S}$ in terms of its representing map as

$$\mathscr{S}_{\mathbb{S}}\big(\mathrm{N}_{\mathbb{1}_{\mathrm{in}}}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})\big) \subseteq \mathrm{N}_{\mathbb{1}_{\mathrm{in}'}}(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'}). \tag{2.29}$$

Unfortunately, though one may have expected TP^2 condition for supermaps $\mathbb{S}$ to be equivalent to TP condition for their representing maps $\mathscr{S}_{\mathbb{S}}$, this is not the case: indeed, it is clear that a (strictly) necessary TP^2 condition is given by

$$\mathscr{S}_{\mathbb{S}}\big(\mathrm{N}_{\mathbb{1}_{\mathrm{in}}/d_{\mathrm{in}'}}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})\big) \subseteq \mathrm{N}_{1}(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'}), \tag{2.30}$$

and a (strictly) sufficient one is given by

$$\mathscr{S}_{\mathbb{S}}\big(\mathrm{N}_{1}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})\big) \subseteq \mathrm{N}_{\mathbb{1}_{\mathrm{in}'}/d_{\mathrm{in}}}(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'}). \tag{2.31}$$

This proves that TP^2 condition has nothing to do with TP condition, which me may write as

$$\mathscr{S}_{\mathbb{S}}\big(\mathrm{N}_{1}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})\big) \subseteq \mathrm{N}_{1}(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'}). \tag{2.32}$$

In fact, the normalization condition in Eq. (2.29) involves the partial trace, instead of symply the trace, as in Eq. (2.32).

The following Theorem succeeds in providing a useful characterization of TP^2 supermaps.

Theorem 2.6 (Characterization of TP^2 Supermaps) *Let $\mathbb{S}$ be a supermap. Then, $\mathbb{S}$ is TP^2 if and only if there exists a* unital *linear map $\mathscr{E}_{\mathbb{S}} : \mathcal{B}(\mathcal{H}_{\mathrm{in}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{in}'})$ such that*

$$\mathrm{Tr}_{\mathrm{out}'}[\mathscr{S}_{\mathbb{S}}(R)] = \mathscr{E}_{\mathbb{S}}(\mathrm{Tr}_{\mathrm{out}}[R]) \tag{2.33}$$

for all $R \in \mathcal{B}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$.

Proof In the present proof, for all sets A and B that allow us to do so, we will denote by $A \pm B$ the set defined by

$$A \pm B = \{a \pm b \mid (a,b) \in A \times B\}. \tag{2.34}$$

Using this notation, we have

$$\begin{aligned} \mathrm{N}_{\mathbb{1}_{\mathrm{in}}}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}) &= \{R \mid \mathrm{Tr}_{\mathrm{out}}[R] = \mathbb{1}_{\mathrm{in}}\} = \\ &= \{\overline{P}\} + \{Q \mid \mathrm{Tr}_{\mathrm{out}}[Q] = \mathbb{1}_{\mathrm{in}} - \mathrm{Tr}_{\mathrm{out}}[\overline{P}]\} = \\ &= \{\overline{P}\} + \mathrm{N}_{\mathbb{1}_{\mathrm{in}} - \mathrm{Tr}_{\mathrm{out}}[\overline{P}]}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}) \end{aligned} \tag{2.35}$$

for all $\overline{P} \in \mathcal{B}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$. Then, thanks to the linearity of the partial trace, we conclude that

$$\mathrm{N}_{\mathbb{1}_{\text{in}}}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}) = \{\overline{P}\} + \mathrm{N}_{\mathbb{1}_{\text{in}}}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}) - \mathrm{N}_{\mathrm{Tr}_{\text{out}}[\overline{P}]}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}) \quad (2.36)$$

and that a similar result holds in $\mathcal{H}_{\text{out}'} \otimes \mathcal{H}_{\text{in}'}$, namely

$$\mathrm{N}_{\mathbb{1}_{\text{in}'}}(\mathcal{H}_{\text{out}'} \otimes \mathcal{H}_{\text{in}'}) = \{\mathscr{S}_{\mathbb{S}}(\overline{P})\} + \mathrm{N}_{\mathbb{1}_{\text{in}'}}(\mathcal{H}_{\text{out}'} \otimes \mathcal{H}_{\text{in}'}) - \mathrm{N}_{\mathrm{Tr}_{\text{out}'}[\mathscr{S}_{\mathbb{S}}(\overline{P})]}(\mathcal{H}_{\text{out}'} \otimes \mathcal{H}_{\text{in}'}) \quad (2.37)$$

for all $\overline{P} \in \mathcal{B}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$. Thus, substituting in Eq. (2.29), we have obtained that $\mathbb{S}$ is TP^2 if and only if

$$\begin{aligned} &\mathscr{S}_{\mathbb{S}}\Big(\{\overline{P}\} + \mathrm{N}_{\mathbb{1}_{\text{in}}}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}) - \mathrm{N}_{\mathrm{Tr}_{\text{out}}[\overline{P}]}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})\Big) \subseteq \\ &\quad \subseteq \{\mathscr{S}_{\mathbb{S}}(\overline{P})\} + \mathrm{N}_{\mathbb{1}_{\text{in}'}}(\mathcal{H}_{\text{out}'} \otimes \mathcal{H}_{\text{in}'}) - \mathrm{N}_{\mathrm{Tr}_{\text{out}'}[\mathscr{S}_{\mathbb{S}}(\overline{P})]}(\mathcal{H}_{\text{out}'} \otimes \mathcal{H}_{\text{in}'}), \end{aligned} \quad (2.38)$$

which, by the linearity of $\mathscr{S}_{\mathbb{S}}$, is equivalent to

$$\begin{aligned} &\mathscr{S}_{\mathbb{S}}\Big(\mathrm{N}_{\mathbb{1}_{\text{in}}}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})\Big) - \mathscr{S}_{\mathbb{S}}\Big(\mathrm{N}_{\mathrm{Tr}_{\text{out}}[\overline{P}]}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})\Big) \subseteq \\ &\quad \subseteq \mathrm{N}_{\mathbb{1}_{\text{in}'}}(\mathcal{H}_{\text{out}'} \otimes \mathcal{H}_{\text{in}'}) - \mathrm{N}_{\mathrm{Tr}_{\text{out}'}[\mathscr{S}_{\mathbb{S}}(\overline{P})]}(\mathcal{H}_{\text{out}'} \otimes \mathcal{H}_{\text{in}'}) \end{aligned} \quad (2.39)$$

for all $\overline{P} \in \mathcal{B}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$. Then, by direct comparison with Eq. (2.29), we see that $\mathbb{S}$ is TP^2 if and only if

$$\begin{cases} \mathscr{S}_{\mathbb{S}}\Big(\mathrm{N}_{\mathbb{1}_{\text{in}}}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})\Big) \subseteq \mathrm{N}_{\mathbb{1}_{\text{in}'}}(\mathcal{H}_{\text{out}'} \otimes \mathcal{H}_{\text{in}'}), \\ \mathscr{S}_{\mathbb{S}}\Big(\mathrm{N}_{\mathrm{Tr}_{\text{out}}[\overline{P}]}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})\Big) \subseteq \mathrm{N}_{\mathrm{Tr}_{\text{out}'}[\mathscr{S}_{\mathbb{S}}(\overline{P})]}(\mathcal{H}_{\text{out}'} \otimes \mathcal{H}_{\text{in}'}) \quad \forall\, \overline{P}, \end{cases} \quad (2.40)$$

which may be expanded as follows:

$$\begin{cases} \mathrm{Tr}_{\text{out}'}[\mathscr{S}_{\mathbb{S}}(R)] = \mathbb{1}_{\text{in}'} \quad \forall\, R \mid \mathrm{Tr}_{\text{out}}[R] = \mathbb{1}_{\text{in}}, \\ \mathrm{Tr}_{\text{out}'}[\mathscr{S}_{\mathbb{S}}(P)] = \mathrm{Tr}_{\text{out}'}[\mathscr{S}_{\mathbb{S}}(\overline{P})] \quad \forall\, (P, \overline{P}) \mid \mathrm{Tr}_{\text{out}}[P] = \mathrm{Tr}_{\text{out}}[\overline{P}]. \end{cases} \quad (2.41)$$

The latter condition shows that the composite map $\mathrm{Tr}_{\text{out}'} \circ \mathscr{S}_{\mathbb{S}} : \mathcal{B}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}) \to \mathcal{B}(\mathcal{H}_{\text{in}'})$ must depend on $\mathrm{Tr}_{\text{out}}[R]$ only, rather than on R, i.e. there must exist a linear map $\mathscr{E}_{\mathbb{S}} : \mathcal{B}(\mathcal{H}_{\text{in}}) \to \mathcal{B}(\mathcal{H}_{\text{in}'})$ such that

$$\mathrm{Tr}_{\text{out}'}[\mathscr{S}_{\mathbb{S}}(P)] = \mathscr{E}_{\mathbb{S}}(\mathrm{Tr}_{\text{out}}[P]) \quad \forall\, P \in \mathcal{B}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}). \quad (2.42)$$

Furthermore, the former condition may be evidently rewritten in terms of the latter as

$$\mathscr{E}_{\mathbb{S}}(\mathbb{1}_{\text{in}}) = \mathbb{1}_{\text{in}'}, \quad (2.43)$$

which proves the Lemma. ■

Remark 2.5 Let us consider the map $\mathscr{E}_{\mathbb{S}}$ as in Theorem 2.6: as we have already seen in Remark 1.11, any map is TP iff, in the Heisenberg picture, it is unital. Then we realize that the Heisenberg-conjugate map $\mathscr{E}_{\mathbb{S}}^{\top}$ is TP if and only if $\mathscr{E}_{\mathbb{S}}$ is unital. So, Theorem 2.6 may be restated as follows: Any supermap $\mathbb{S}$ is TP^2 if and only if there exists a TP map $\mathscr{E}_{\mathbb{S}}^{\top} : \mathcal{T}(\mathcal{H}_{\mathrm{in}'}) \to \mathcal{T}(\mathcal{H}_{\mathrm{in}})$ such that its Heisenberg-conjugate map, $\mathscr{E}_{\mathbb{S}}$, satisfies Eq. (2.33). This latter condition may be rewritten in terms of $\mathscr{E}_{\mathbb{S}}^{\top}$ as well: in fact, from Remark 1.10 we know that the definition of Heisenberg-conjugate maps is equivalent to condition (1.52). Then, if we fix $A = \mathrm{Tr}_{\mathrm{out}}[R]$ for some $R \in \mathcal{B}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$, we obtain the following scheme of equivalences for all $A' \in \mathcal{T}(\mathcal{H}_{\mathrm{in}'})$:

$$\begin{array}{ccc} \mathrm{Tr}[A'\mathscr{E}_{\mathbb{S}}(\mathrm{Tr}_{\mathrm{out}}[R])] & \overset{(1.52)}{=} & \mathrm{Tr}\left[\left(\mathbb{1}_{\mathrm{out}} \otimes \mathscr{E}_{\mathbb{S}}^{\top}(A')\right)R\right] \\ (2.33) \shortparallel & & \\ \mathrm{Tr}\left[\left(\mathbb{1}_{\mathrm{out}'} \otimes A'\right)\mathscr{S}_{\mathbb{S}}(R)\right] & \overset{(1.52)}{=} & \mathrm{Tr}\left[\mathscr{S}_{\mathbb{S}}^{\top}(\mathbb{1}_{\mathrm{out}'} \otimes A')R\right] \end{array} \tag{2.44}$$

Then, requiring Eq. (2.33) to hold for all R is equivalent to requiring

$$\mathscr{S}_{\mathbb{S}}^{\top}(\mathbb{1}_{\mathrm{out}'} \otimes A') = \mathbb{1}_{\mathrm{out}} \otimes \mathscr{E}_{\mathbb{S}}^{\top}(A'). \tag{2.45}$$

Summarizing the above results, we may state the following

Corollary 2.7 (to Theorem 2.6) *Let $\mathbb{S}$ be a supermap. Then, $\mathbb{S}$ is* TP^2 *if and only if there exists a* TP *map $\mathscr{E}_{\mathbb{S}}^{\top} : \mathcal{T}(\mathcal{H}_{\mathrm{in}'}) \to \mathcal{T}(\mathcal{H}_{\mathrm{in}})$ such that Eq. (2.45) holds for all $A' \in \mathcal{T}(\mathcal{H}_{\mathrm{in}'})$ — where $\mathscr{S}_{\mathbb{S}}^{\top}$ is the Heisenberg-conjugate of the representative map $\mathscr{S}_{\mathbb{S}}$.*

This result does not provide more insight into the problem of normalization than Theorem 2.6. Nevertheless, it will be of great help when we study the physical realization of supermaps in Section 2.4. ▲

Remark 2.6 For every TP^2 supermap $\mathbb{S}$, the map $\mathscr{E}_{\mathbb{S}}$ satisfying Eq. (2.33) has a corresponding Choi operator given by

$$\begin{aligned} R_{\mathscr{E}_{\mathbb{S}}} &= \sum_{i,j=1}^{d_{\mathrm{in}}} \mathscr{E}_{\mathbb{S}}(|i\rangle_{\mathrm{in}}\langle j|) \otimes |i\rangle_{\mathrm{in}}\langle j| = \\ &= \sum_{i,j=1}^{d_{\mathrm{in}}} \mathrm{Tr}_{\mathrm{out}'}\left[\mathscr{S}_{\mathbb{S}}\left(\frac{\mathbb{1}_{\mathrm{out}}}{d_{\mathrm{out}}} \otimes |i\rangle_{\mathrm{in}}\langle j|\right)\right] \otimes |i\rangle_{\mathrm{in}}\langle j| = \\ &= \sum_{i,j=1}^{d_{\mathrm{in}}} \mathrm{Tr}_{\mathrm{out}',\mathrm{out},\mathrm{in}}\left[\left(\mathbb{1}_{\mathrm{out}'} \otimes \mathbb{1}_{\mathrm{in}'} \otimes \frac{\mathbb{1}_{\mathrm{out}}}{d_{\mathrm{out}}} \otimes |j\rangle_{\mathrm{in}}\langle i|\right) R_{\mathbb{S}}\right] \otimes |i\rangle_{\mathrm{in}}\langle j| = \\ &= \frac{1}{d_{\mathrm{out}}}\mathrm{Tr}_{\mathrm{out}',\mathrm{out}}[R_{\mathbb{S}}]. \end{aligned} \tag{2.46}$$

This shows that, if $\mathbb{S}$ is $\mathrm{C^2P^2}$ as well, then $\mathscr{E}_{\mathbb{S}}$ is CP, whilst the opposite is no more true in general. ▲

Remark 2.7 Since we have a $\mathrm{C^2P^2}$ condition which is expressed in terms of the Choi operator $R_{\mathbb{S}}$, we may wonder whether there is a way to restate Theorem 2.6 in terms of $R_{\mathbb{S}}$. It is easy to check that $R_{\mathbb{S}}$ is normalized (namely, $\mathbb{S}$ is TP^2) if and only if there exists a $\tilde{R} \in \mathcal{B}(\mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{in}})$ such that

$$\begin{cases} \mathrm{Tr}_{\mathrm{in}}[\tilde{R}] = \mathbb{1}_{\mathrm{in}'}, \\ \mathrm{Tr}_{\mathrm{out,in}}\left[(\mathbb{1}_{\mathrm{in}'} \otimes R^\top)\mathrm{Tr}_{\mathrm{out}'}[R_{\mathbb{S}}]\right] = \mathrm{Tr}_{\mathrm{out,in}}\left[(\mathbb{1}_{\mathrm{in}'} \otimes R^\top)(\tilde{R} \otimes \mathbb{1}_{\mathrm{out}})\right] \end{cases} \tag{2.47}$$

for all $R \in \mathcal{B}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$. This leads us to state the following

Corollary 2.8 (to Theorem 2.6) *Let $\mathbb{S}$ be a supermap. Then, $\mathbb{S}$ is TP^2 if and only if its Choi operator satisfies*

$$\mathrm{Tr}_{\mathrm{out}'}[R_{\mathbb{S}}] = \tilde{R} \otimes \mathbb{1}_{\mathrm{out}} \tag{2.48}$$

for some $R \in \mathcal{B}(\mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{in}})$ such that $\mathrm{Tr}_{\mathrm{in}}[\tilde{R}] = \mathbb{1}_{\mathrm{in}'}$.

In the following, we will denote with $\Theta(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$ the set of normalized Choi operators, namely

$$\begin{aligned} \Theta(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}) \doteq \Big\{ & R \in \mathcal{B}(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}) \,\Big| \\ & \Big|\, \exists\, R \in \mathcal{B}(\mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{in}}) \text{ s.t. conditions (2.47) hold} \Big\}, \end{aligned} \tag{2.49}$$

and of course we have the Choi isomorphism

$$\mathrm{TP}^2(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}; \mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'}) \cong \Theta(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}). \tag{2.50}$$

▲

2.2.4 More Results on TP-Preserving Supermaps

In the present Subsection, we prove two results regarding TP^2 supermaps that were previously anticipated.

Lemma 2.9 (QCP **and** TP^2 **Supermaps)** *All Quantum Channel-Preserving supermaps also preserve the Trace-Preserving property of maps.*

Proof As in Lemma 1.1, we will give the proof by contradiction. So, let $\mathbb{S}$ be QCP, and let us suppose that there exists a TP map $\mathscr{C}$ which is not a Quantum Channel (i.e. it is not CP) such that the output map $\mathbb{S}(\mathscr{C})$ is no more TP. Exploiting Choi isomorphism, this is equivalent to the existence of an operator $O \in \mathrm{N}_{\mathbb{1}_{\mathrm{in}}}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}) \setminus \Omega(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$ such that $\mathscr{S}_{\mathbb{S}}(O) \notin \mathrm{N}_{\mathbb{1}_{\mathrm{in}'}}(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'})$. Furthermore, let us consider the line $\{\overline{O}_r \mid r \in \mathbb{R}\}$ in the hyperplane $\mathrm{N}_{\mathbb{1}_{\mathrm{in}}}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$ parametrized by

$$\overline{O}_r = rO + (1-r)R, \quad r \in \mathbb{R}, \tag{2.51}$$

where R is some Choi operator corresponding to a full-rank, non-extremal Quantum Channel. $\overline{O}_r$ is easily checked to be in $\mathrm{N}_{\mathbb{1}_{\mathrm{in}}}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$, for all r; furthermore, we have

$$\begin{aligned} r\mathrm{Tr}_{\mathrm{out}'}[\mathscr{S}_{\mathbb{S}}(O)] &= \mathrm{Tr}_{\mathrm{out}'}[\mathscr{S}_{\mathbb{S}}(\overline{O}_r)] - (1-r)\mathrm{Tr}_{\mathrm{out}'}[\mathscr{S}_{\mathbb{S}}(R)] = \\ &= \mathrm{Tr}_{\mathrm{out}'}[\mathscr{S}_{\mathbb{S}}(\overline{O}_r)] + (r-1)\mathbb{1}_{\mathrm{in}'}, \end{aligned} \tag{2.52}$$

thanks to the hypothesis that $\mathbb{S}$ is QCP. So, thanks to the linearity of $\mathscr{S}_{\mathbb{S}}$ (and to that of the trace), the hypothesis that $\mathrm{Tr}_{\mathrm{out}'}[\mathscr{S}_{\mathbb{S}}(O)] \neq \mathbb{1}_{\mathrm{in}'}$ yields

$$\mathrm{Tr}_{\mathrm{out}'}[\mathscr{S}_{\mathbb{S}}(\overline{O}_r)] \neq \mathbb{1}_{\mathrm{in}'} \quad \forall\, r \in \mathbb{R} \setminus \{0\}. \tag{2.53}$$

Now, since $\overline{O}_0 = R$, and since R is non-extremal in $[\mathrm{N}_{\mathbb{1}_{\mathrm{in}}} \cap \Omega](\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$, we can always find an $\varepsilon > 0$ such that $\overline{O}_r$ is still an element of $[\mathrm{N}_{\mathbb{1}_{\mathrm{in}}} \cap \Omega](\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$ for all $r \in [-\varepsilon, +\varepsilon]$: this means that

$$\mathrm{Tr}_{\mathrm{out}'}[\mathscr{S}_{\mathbb{S}}(\overline{O}_r)] = \mathbb{1}_{\mathrm{in}'} \quad \forall\, r \in [-\varepsilon, +\varepsilon], \tag{2.54}$$

which contradicts Eq. (2.53). ∎

Lemma 2.10 (Extensions of TP^2 Supermaps) *All TP-Preserving supermaps are Completely-TP Preserving as well, namely*

$$\mathbb{S} \otimes \mathbb{I} \in \mathrm{TP}^2(\mathcal{H}_{\mathrm{in}} \otimes \mathcal{K}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}} \otimes \mathcal{K}_{\mathrm{out}}; \mathcal{H}_{\mathrm{in}'} \otimes \mathcal{K}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}'} \otimes \mathcal{K}_{\mathrm{out}}) \tag{2.55}$$

for all $\mathbb{S} \in \mathrm{TP}^2(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}; \mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'})$.

Proof Let us use Theorem 2.6: then we see that $\mathbb{S} \otimes \mathbb{I}$ is TP^2 if and only if there exists a unital linear map $\underline{\mathscr{E}}_{\mathbb{S}\otimes\mathbb{I}} : \mathcal{B}(\mathcal{H}_{\mathrm{in}} \otimes \mathcal{K}_{\mathrm{in}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{in}'} \otimes \mathcal{K}_{\mathrm{in}})$ such that

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{out}'},\mathcal{K}_{\mathrm{out}}}[\mathscr{S}_{\mathbb{S}\otimes\mathbb{I}}(\underline{R})] = \underline{\mathscr{E}}_{\mathbb{S}\otimes\mathbb{I}}\left(\mathrm{Tr}_{\mathcal{H}_{\mathrm{out}},\mathcal{K}_{\mathrm{out}}}[\underline{R}]\right) \tag{2.56}$$

for all $\underline{R} \in \mathcal{B}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{K}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}} \otimes \mathcal{K}_{\mathrm{in}})$.

Of course, since $\mathbb{S}$ is TP^2, we can find a unital linear map $\mathscr{E}_{\mathbb{S}} : \mathcal{B}(\mathcal{H}_{\text{in}}) \to \mathcal{B}(\mathcal{H}_{\text{in}'})$ such that

$$\mathrm{Tr}_{\mathcal{H}_{\text{out}'}}[\mathscr{S}_{\mathbb{S}}(R)] = \mathscr{E}_{\mathbb{S}}\left(\mathrm{Tr}_{\mathcal{H}_{\text{out}}}[R]\right) \tag{2.57}$$

for all $R \in \mathcal{B}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$. Now, using Lemma 2.2, we may write

$$\mathrm{Tr}_{\mathcal{H}_{\text{out}'},\mathcal{K}_{\text{out}}}[\mathscr{S}_{\mathbb{S}\otimes\mathbb{I}}(\underline{R})] = \sum_{i=1}^{r_{\underline{R}}} \mathrm{Tr}_{\mathcal{H}_{\text{out}'}}[\mathscr{S}_{\mathbb{S}}(H_i)] \otimes \mathrm{Tr}_{\mathcal{K}_{\text{out}}}[K_i] \tag{2.58}$$

where $\underline{R} = \sum_i H_i \otimes K_i$ is the Schmidt decomposition for $\underline{R}$, with $H_i \in \mathcal{B}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$ and $K_i \in \mathcal{B}(\mathcal{K}_{\text{out}} \otimes \mathcal{K}_{\text{in}})$. So we have

$$\mathrm{Tr}_{\mathcal{H}_{\text{out}'},\mathcal{K}_{\text{out}}}[\mathscr{S}_{\mathbb{S}\otimes\mathbb{I}}(\underline{R})] = \sum_{i=1}^{r_{\underline{R}}} \mathscr{E}_{\mathbb{S}}\left(\mathrm{Tr}_{\mathcal{H}_{\text{out}}}[H_i]\right) \otimes \mathrm{Tr}_{\mathcal{K}_{\text{out}}}[K_i] \tag{2.59}$$

and it is sufficient to define

$$\underline{\mathscr{E}}_{\mathbb{S}\otimes\mathbb{I}}\left(\mathrm{Tr}_{\mathcal{H}_{\text{out}},\mathcal{K}_{\text{out}}}[\underline{R}]\right) \doteq \mathscr{E}_{\mathbb{S}} \otimes \mathscr{I}_{\mathcal{K}_{\text{in}}} \tag{2.60}$$

to obtain Eq. (2.57). Furthermore, it should be clear that $\underline{\mathscr{E}}_{\mathbb{S}\otimes\mathbb{I}}$ is unital if and only if $\mathscr{E}_{\mathbb{S}}$ is so. This concludes the proof. ■

2.3 Covariant Supermaps

In the present Section we are particularly interested in those supermaps such that the unitary action of some group compact or finite $\mathbf{G}$ on the the input space $\mathcal{H}_{\text{in}}$ of the *input map* (and/or on its output space $\mathcal{H}_{\text{out}}$) is equivalent to the action of $\mathbf{G}$ on the input space $\mathcal{H}_{\text{in}'}$ of its *output map* (and/or on its output space $\mathcal{H}_{\text{out}'}$). The reason for our interest is that there are some relevant cases in which it is physically meaningfull to deal only with supermaps exhibiting such symmetries.

Note that a few basic notions of groups and representation theory are presented in Appendix A.

2.3.1 Preliminary Definitions

In order to define covariant supermaps in an abstract way, let us introduce a compact group $\mathbf{G}$, and four unitary representations $(U, \mathcal{H}_{\text{in}})$, $(V, \mathcal{H}_{\text{out}})$,

$(U', \mathcal{H}_{\mathrm{in}'})$ and $(V', \mathcal{H}_{\mathrm{out}'})$. Furthermore, let us define representations $\mathbb{U}$ and $\mathbb{V}$ on the Hilbert spaces of input maps via

$$\begin{cases} \mathbb{U}_g(\mathscr{C}) \doteq \mathscr{C} \circ [U_g \bullet U_g^\dagger] \\ \mathbb{V}_g(\mathscr{C}) \doteq [V_g \bullet V_g^\dagger] \circ \mathscr{C} \end{cases} \quad \forall\, g \in \mathbf{G}, \tag{2.61}$$

and representations $\mathbb{U}'$, $\mathbb{V}'$ on the spaces of output maps via

$$\begin{cases} \mathbb{U}'_g(\mathscr{C}') \doteq \mathscr{C}' \circ [U'_g \bullet U'^\dagger_g] \\ \mathbb{V}'_g(\mathscr{C}') \doteq [V'_g \bullet V'^\dagger_g] \circ \mathscr{C}' \end{cases} \quad \forall\, g \in \mathbf{G}. \tag{2.62}$$

Now, we are ready to state

Definition 2.7 (Covariant Supermap) *Let $\mathbb{S}$ be a supermap. We will say that $\mathbb{S}$ is* **G**-covariant on its input spaces *when*

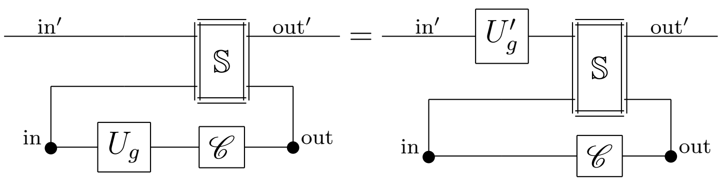
$$\mathbb{S}\mathbb{U}_g = \mathbb{U}'_g\mathbb{S} \quad \forall\, g \in \mathbf{G}, \tag{2.63}$$

and that it is **G**-covariant on its output spaces *when*

$$\mathbb{S}\mathbb{V}_g = \mathbb{V}'_g\mathbb{S} \quad \forall\, g \in \mathbf{G}. \tag{2.64}$$

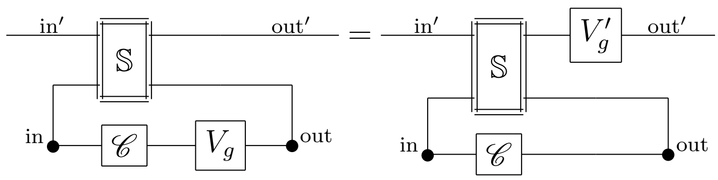
Finally, we will say that $\mathbb{S}$ is two-fold **G**-covariant *when it is covariant respect to* **G** *on both the input and the output spaces.*

Remark 2.8 It is straightforward to realize that, diagramatically, the **G**-covariance condition on input spaces (2.63) reads



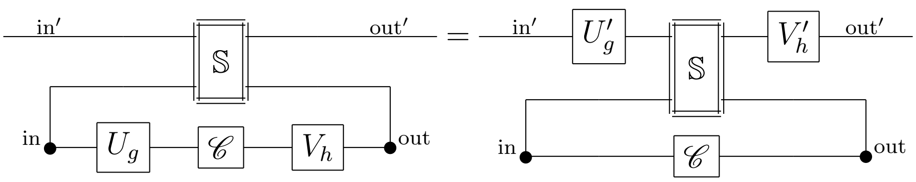
$\forall\, g \in \mathbf{G}$, (2.65)

whilst the one on output spaces (2.64) reads



$\forall\, g \in \mathbf{G}$. (2.66)

Of course, then, two-fold **G**-covariance is equivalent to



(2.67)

for all (g, h) in $\mathbf{G} \times \mathbf{G}$. ▲

Remark 2.9 Notice that, since covariance conditions are linear, then the three sets of input-covariant, output-covariant and two-fold-covariant supermaps are affine spaces. ▲

2.3.2 Characterization of Covariant Supermaps

In order to be able to implement covariant supermaps, we need to restate Definition 2.7 in terms of the Choi operators of supermaps. This is the goal of the following Lemma[2].

Lemma 2.11 (Characterization of Covariant Supermaps) *Let $\mathbb{S}$ be a linear supermap; then, $\mathbb{S}$ is $\mathbf{G}$-covariant on its input spaces iff its Choi operator satisfies*

$$\left[R_{\mathbb{S}}^*, \mathbb{1}_{\mathrm{out}'} \otimes U_g' \otimes \mathbb{1}_{\mathrm{out}} \otimes U_g^*\right] = 0 \quad \forall\, g \in \mathbf{G}, \tag{2.68}$$

and it is $\mathbf{G}$-covariant on its output spaces iff

$$\left[R_{\mathbb{S}}, V_g' \otimes \mathbb{1}_{\mathrm{in}'} \otimes V_g^* \otimes \mathbb{1}_{\mathrm{in}}\right] = 0 \quad \forall\, g \in \mathbf{G}. \tag{2.69}$$

Proof As a first step, we shall prove that the Choi operator corresponding to the composition of supermaps $\mathbb{V}_{h'}'\mathbb{U}_{g'}'\mathbb{S}\mathbb{V}_h\mathbb{U}_g$ is explicitly given by

$$R_{[\mathbb{V}_{h'}'\mathbb{U}_{g'}'\mathbb{S}\mathbb{V}_h\mathbb{U}_g]} = \left[V_{h'}' \otimes {U_{g'}'}^{\top} \otimes V_h^{\top} \otimes U_g\right] R_{\mathbb{S}} \left[V_{h'}' \otimes {U_{g'}'}^{\top} \otimes V_h^{\top} \otimes U_g\right]^{\dagger}. \tag{2.70}$$

[2]We remind that Choi operators are linear operators on $\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}$.

Indeed, the proof is a consequence of the inverse isomorphism formula for sumermaps, Eq. (2.11): for all $A' \in \mathcal{T}(\mathcal{H}_{\mathrm{in}'})$ we have

$$
\begin{aligned}
&\Big[[\mathbb{V}'_{h'}\mathbb{U}'_{g'}\mathbb{S}\mathbb{V}_h\mathbb{U}_g](\mathscr{C})\Big](A') = \\
&= V'_{h'}\left\{\Big[\mathbb{S}\big((V_h \bullet V_h^\dagger)\mathscr{C}(U_g \bullet U_g^\dagger)\big)\Big](U'_{g'}A'U'^{\dagger}_{g'})\right\}V'^{\dagger}_{h'} = \\
&= V'_{h'}\ \mathrm{Tr}_{\mathrm{in}',\mathrm{out},\mathrm{in}}\left[\Big(\mathbb{1}_{\mathrm{out}'} \otimes (U'_{g'}A'U'^{\dagger}_{g'})^\top \otimes R^\top_{[(V_h\bullet V_h^\dagger)\mathscr{C}(U_g\bullet U_g^\dagger)]}\Big)R_{\mathbb{S}}\right]\ V'^{\dagger}_{h'} = \\
&= V'_{h'}\ \mathrm{Tr}_{\mathrm{in}',\mathrm{out},\mathrm{in}}\Big[\Big(\mathbb{1}_{\mathrm{out}'} \otimes U'^{*}_{g'}A'^\top U'^{\top}_{g'} \otimes \\
&\qquad\qquad \otimes \big[V_h \otimes U_g^\top\big]^* R^\top_{\mathscr{C}}\big[V_h \otimes U_g^\top\big]^\top\Big)R_{\mathbb{S}}\Big]\ V'^{\dagger}_{h'} = \\
&= \mathrm{Tr}_{\mathrm{in}',\mathrm{out},\mathrm{in}}\left[\Big(\mathbb{1}_{\mathrm{out}'} \otimes A'^\top \otimes R^\top_{\mathscr{C}}\Big)\Big[V'_{h'} \otimes U'^{\top}_{g'} \otimes V_h^\top \otimes U_g\Big]R_{\mathbb{S}}\Big[\dots\Big]^\dagger\right],
\end{aligned}
\tag{2.71}
$$

where we have used the additional formula

$$
R_{(V_h\bullet V_h^\dagger)\circ\mathscr{C}\circ(U_g\bullet U_g^\dagger)} = \Big[V_h \otimes U_g^\top\Big]R_{\mathscr{C}}\Big[V_h \otimes U_g^\top\Big]^\dagger, \tag{2.72}
$$

that can be proved by

$$
\begin{aligned}
&[(V_h \bullet V_h^\dagger) \circ \mathscr{C} \circ (U_g \bullet U_g^\dagger)](A) = \\
&\quad = V_h\mathscr{C}(U_gAU_g^\dagger)V_h^\dagger = \\
&\quad = V_h\mathrm{Tr}_{\mathrm{in}}\left[(\mathbb{1}_{\mathrm{out}} \otimes U_g^*A^\top U_g^\top)R_{\mathscr{C}}\right]V_h^\dagger = \\
&\quad = V_h\mathrm{Tr}_{\mathrm{in}}\left[(\mathbb{1}_{\mathrm{out}} \otimes U_g^*)(\mathbb{1}_{\mathrm{out}} \otimes A^\top)(\mathbb{1}_{\mathrm{out}} \otimes U_g^\top)R_{\mathscr{C}}\right]V_h^\dagger = \\
&\quad = \mathrm{Tr}_{\mathrm{in}}\left[(\mathbb{1}_{\mathrm{out}} \otimes A^\top)\Big[V_h \otimes U_g^\top\Big]R_{\mathscr{C}}\Big[V_h \otimes U_g^\top\Big]^\dagger\right]
\end{aligned}
\tag{2.73}
$$

for all $A \in \mathcal{T}(\mathcal{H}_{\mathrm{in}})$.

Now, of course $\mathbb{S}$ is $\mathbf{G}$-covariant on its input spaces iff

$$
R_{\mathbb{S}} = R_{\mathbb{U}'_{g^{-1}}\mathbb{S}\mathbb{U}_g} \quad \forall\, g \in \mathbf{G}, \tag{2.74}
$$

and using Eq. (2.70) yields Eq. (2.68). Similarly, $\mathbb{S}$ is $\mathbf{G}$-covariant on its output spaces iff

$$
R_{\mathbb{S}} = R_{\mathbb{V}'_g\mathbb{S}\mathbb{V}_{g^{-1}}} \quad \forall\, g \in \mathbf{G}, \tag{2.75}
$$

and using Eq. (2.70) yields Eq. (2.69). ■

Remark 2.10 Lemma 2.11 trivially implies that $\mathbb{S}$ is *two-fold* $\mathbf{G}$-covariant if and only if its Choi operator satisfies

$$
\Big[R_{\mathbb{S}}, V'_h \otimes U'^{*}_g \otimes V_h^* \otimes U_g\Big] = 0 \quad \forall\, (g,h) \in \mathbf{G} \times \mathbf{G}. \tag{2.76}
$$

Furthermore, it is easy to show that Eqs. (2.68, 2.69) are respectively equivalent to

$$\int_{\mathbf{G}} \mathrm{d}g \left[\mathbb{1}_{\mathrm{out}'} \otimes U'_g \otimes \mathbb{1}_{\mathrm{out}} \otimes U^*_g \right] R^*_{\mathbb{S}} \left[\mathbb{1}_{\mathrm{out}'} \otimes U'_g \otimes \mathbb{1}_{\mathrm{out}} \otimes U^*_g \right]^\dagger = R^*_{\mathbb{S}}, \tag{2.77}$$

and to

$$\int_{\mathbf{G}} \mathrm{d}g \left[V'_g \otimes \mathbb{1}_{\mathrm{in}'} \otimes V^*_g \otimes \mathbb{1}_{\mathrm{in}} \right] R_{\mathbb{S}} \left[V'_g \otimes \mathbb{1}_{\mathrm{in}'} \otimes V^*_g \otimes \mathbb{1}_{\mathrm{in}} \right]^\dagger = R_{\mathbb{S}}. \tag{2.78}$$

Indeed, the fact that they are necessary conditions for the previous ones to be true is trivial: to see that they are also sufficient, let us consider, for example, Eq. (2.77). Then, applying $[\mathbb{1}_{\mathrm{out}'} \otimes {U'_{g'}}^* \otimes \mathbb{1}_{\mathrm{out}} \otimes U_{g'}] \bullet [\mathbb{1}_{\mathrm{out}'} \otimes {U'_{g'}}^* \otimes \mathbb{1}_{\mathrm{out}} \otimes U_{g'}]^\dagger$ on both of its sides simply sums up to performing a change of variables in the left-hand side ($g \mapsto g'g$), so that Eq. (2.68) is obtained.

This also implies that $\mathbb{S}$ is two-fold $\mathbf{G}$-covariant if and only if

$$\int_{\mathbf{G}} \mathrm{d}g \int_{\mathbf{G}} \mathrm{d}h \left[V'_h \otimes {U'_g}^* \otimes V^*_h \otimes U_g \right] R_{\mathbb{S}} \left[V'_h \otimes {U'_g}^* \otimes V^*_h \otimes U_g \right]^\dagger = R_{\mathbb{S}}. \tag{2.79}$$

▲

2.3.3 Isotypic Decomposition of Covariant Supermaps

Lemma 2.11 shows that the $\mathbf{G}$-covariance of supermaps is equivalent to the $\mathbf{G}$-commutation of its Choi operators. Clearly, then, Theorem A.2 provides a convenient characterization of covariant supermaps, that we state in the following

Theorem 2.12 (Characterization of Covariant Supermaps) *Let the two unitary representations* $(V' \otimes V^*, \mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{out}})$ *and* $(U' \otimes U^*, \mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{in}})$ *of* $\mathbf{G}$ *admit the following isotypic decompositions:*

$$\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{out}} \cong \bigoplus_{\mu=1}^{|\mathrm{Irrep}(V' \otimes V^*)|} \bigoplus_{i=1}^{m_\mu} \mathcal{H}_i^{(\mu)}, \tag{2.80}$$

$$\mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{in}} \cong \bigoplus_{\nu=1}^{|\mathrm{Irrep}(U' \otimes U^*)|} \bigoplus_{k=1}^{m'_\nu} \mathcal{H}_k^{(\nu)\prime}. \tag{2.81}$$

Then, $\mathbb{S}$ is covariant on its input spaces, on its output ones, on both, respectively if and only if its Choi operator admits the following decomposition:

$$R_{\mathbb{S}} = \sum_{\nu=1}^{|\mathrm{Irrep}(U'\otimes U^*)|} \sum_{k,l=1}^{m'_\nu} R^{(\nu)}_{\mathbb{S};k,l} \otimes T^{(\nu)\prime *}_{l,k}, \tag{2.82}$$

$$R_{\mathbb{S}} = \sum_{\mu=1}^{|\mathrm{Irrep}(V'\otimes V^*)|} \sum_{i,j=1}^{m_\mu} T^{(\mu)}_{j,i} \otimes R^{(\mu)}_{\mathbb{S};i,j}{}', \tag{2.83}$$

$$R_{\mathbb{S}} = \sum_{\nu=1}^{|\mathrm{Irrep}(U'\otimes U^*)|} \sum_{\mu=1}^{|\mathrm{Irrep}(V'\otimes V^*)|} \sum_{i,j=1}^{m_\mu} \sum_{k,l=1}^{m'_\nu} r^{(\mu,\nu)}_{\mathbb{S};i,j,k,l} T^{(\mu)}_{j,i} \otimes T^{(\nu)\prime *}_{l,k}, \tag{2.84}$$

where $T^{(\mu)}_{j,i} : \mathcal{H}^{(\mu)}_i \to \mathcal{H}^{(\mu)}_j$ (respectively, $T^{(\nu)}_{l,k}{}' : \mathcal{H}^{(\nu)}_k{}' \to \mathcal{H}^{(\nu)}_l{}'$) are isometries between equivalent irreducible subspaces of $\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{out}}$ (respectively, of $\mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{in}}$).

Proof Let $\{|m\rangle\rangle_{\mathrm{out}',\mathrm{out}}\}$ and $\{|n\rangle\rangle_{\mathrm{in}',\mathrm{in}}\}$ be orthonormal bases for $\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{out}}$ and, respectively, $\mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{in}}$, and let us rewrite the **G**-covariance condition on, say, output spaces (Eq. (2.69)), in such bases:

$$\begin{aligned} 0 &= \left[R_{\mathbb{S}}, V'_g \otimes \mathbb{1}_{\mathrm{in}'} \otimes V^*_g \otimes \mathbb{1}_{\mathrm{in}}\right] = \\ &= \sum_{m,m'=1}^{d_{\mathrm{out}'}\cdot d_{\mathrm{out}}} \sum_{n,n'=1}^{d_{\mathrm{in}'}\cdot d_{\mathrm{in}}} \langle\langle m|\langle\langle n|R_{\mathbb{S}}|m'\rangle\rangle|n'\rangle\rangle \cdot \\ &\qquad \cdot \left[|m\rangle\rangle\langle\langle m|' \otimes |n\rangle\rangle\langle\langle n'|, V'_g \otimes \mathbb{1}_{\mathrm{in}'} \otimes V^*_g \otimes \mathbb{1}_{\mathrm{in}}\right] = \\ &= \sum_{n,n'=1}^{d_{\mathrm{in}'}\cdot d_{\mathrm{in}}} \left[\big(\mathbb{1}_{\mathrm{out}',\mathrm{out}} \otimes \langle\langle n|\big) R_{\mathbb{S}} \big(\mathbb{1}_{\mathrm{out}',\mathrm{out}} \otimes |n'\rangle\rangle\big), V'_g \otimes V^*_g\right] \otimes |n\rangle\rangle\langle\langle n'|, \end{aligned} \tag{2.85}$$

which is equivalent to

$$\left[\big(\mathbb{1}_{\mathrm{out}',\mathrm{out}} \otimes \langle\langle n|\big) R_{\mathbb{S}} \big(\mathbb{1}_{\mathrm{out}',\mathrm{out}} \otimes |n'\rangle\rangle\big), V'_g \otimes V^*_g\right] = 0 \tag{2.86}$$

for all $n, n' = 1, \ldots, d_{\mathrm{in}'} \cdot d_{\mathrm{in}}$. Thus, Eq. (A.13) yields

$$\begin{aligned} &\big(\mathbb{1}_{\mathrm{out}',\mathrm{out}} \otimes \langle\langle n|\big) R_{\mathbb{S}} \big(\mathbb{1}_{\mathrm{out}',\mathrm{out}} \otimes |n'\rangle\rangle\big) = \\ &\quad = \sum_{\mu=1}^{|\mathrm{Irrep}(V'\otimes V^*)|} \sum_{i,j=1}^{m_\mu} \frac{\mathrm{Tr}[T^{(\mu)}_{i,j}(\mathbb{1} \otimes \langle\langle n|)R_{\mathbb{S}}(\mathbb{1} \otimes |n'\rangle\rangle)]}{d_\mu} T^{(\mu)}_{j,i}. \end{aligned} \tag{2.87}$$

Finally, since of course we have

$$R_{\mathbb{S}} = \sum_{n,n'=1}^{d_{\mathrm{out}'}\cdot d_{\mathrm{out}}} \big(\mathbb{1}_{\mathrm{out}',\mathrm{out}} \otimes \langle\langle n|\big) R_{\mathbb{S}} \big(\mathbb{1}_{\mathrm{out}',\mathrm{out}} \otimes |n'\rangle\rangle\big) \otimes |n\rangle\rangle\langle\langle n'|, \tag{2.88}$$

we have obtained

$$\begin{aligned} R_{\mathbb{S}} &= \sum_{\mu=1}^{|\mathrm{Irrep}(V'\otimes V^*)|} \sum_{i,j=1}^{m_\mu} T_{j,i}^{(\mu)} \otimes \\ &\otimes \sum_{n,n'=1}^{d_{\mathrm{out}'}\cdot d_{\mathrm{out}}} \frac{\mathrm{Tr}[T_{i,j}^{(\mu)}(\mathbb{1}\otimes \langle\langle n|)R_{\mathbb{S}}(\mathbb{1}\otimes |n'\rangle\rangle)]}{d_\mu} |n\rangle\rangle\langle\langle n'| = \\ &= \sum_{\mu=1}^{|\mathrm{Irrep}(V'\otimes V^*)|} \sum_{i,j=1}^{m_\mu} T_{j,i}^{(\mu)} \otimes \frac{\mathrm{Tr}_{\mathrm{out}',\mathrm{out}}[(T_{i,j}^{(\mu)}\otimes \mathbb{1}_{\mathrm{in}',\mathrm{in}})R_{\mathbb{S}}]}{d_\mu}, \end{aligned} \tag{2.89}$$

i.e. we have proved decomposition (2.83) with $R_{\mathbb{S};i,j}^{(\mu)}{}'$ being defined by

$$R_{\mathbb{S};i,j}^{(\mu)}{}' \doteq \frac{\mathrm{Tr}_{\mathrm{out}',\mathrm{out}}[(T_{i,j}^{(\mu)}\otimes \mathbb{1}_{\mathrm{in}',\mathrm{in}})R_{\mathbb{S}}]}{d_\mu}. \tag{2.90}$$

Similarly, one obtains

$$R_{\mathbb{S};k,l}^{(\nu)} = \frac{\mathrm{Tr}_{\mathrm{in}',\mathrm{in}}[(\mathbb{1}_{\mathrm{out}',\mathrm{out}}\otimes T_{k,l}^{(\nu)\prime *})R_{\mathbb{S}}]}{d'_\nu} \tag{2.91}$$

for Eq. (2.82), and

$$r_{\mathbb{S};i,j,k,l}^{(\mu,\nu)} = \frac{\mathrm{Tr}[(T_{i,j}^{(\mu)}\otimes T_{k,l}^{(\nu)\prime *})R_{\mathbb{S}}]}{d_\mu \cdot d'_\nu} \tag{2.92}$$

for Eq. (2.84). ■

2.3.4 Normalization of Covariant Supermaps

In the present Subsection, we retain the notation that was previously used: in particular, we will use symbols whose meaning was introduced in Theorem 2.12.

Theorem 2.13 (Normalization of Covariant Supermaps) *Let $\mathbb{S}$ be a* $\mathbf{G}$*-covariant supermap on its output spaces in the form (2.83), and let $(V,\mathcal{H}_{\mathrm{out}})$ be irreducible. Then, $\mathbb{S}$ is* TP^2 *if and only if*

$$\sum_{\mu=1}^{|\mathrm{Irrep}(V'\otimes V^*)|} d_\mu \sum_{i=1}^{m_\mu} \mathrm{Tr}_{\mathrm{in}}[R_{\mathbb{S};i,i}^{(\mu)}{}'] = d_{\mathrm{out}}\mathbb{1}_{\mathrm{in}'}, \tag{2.93}$$

where d_μ is the dimension of the μ-th invariant subspace $\mathcal{H}^{(\mu)}$ as defined in isotypic decomposition (2.80). Furthermore, if $\mathbb{S}$ is $\mathbf{G}$-covariant on its input spaces as well (i.e. it is a two-fold $\mathbf{G}$-covariant supermap in the form (2.84)), then it is TP^2 *if and only if*

$$\sum_{\nu=1}^{|\mathrm{Irrep}(U'\otimes U^*)|}\sum_{\mu=1}^{|\mathrm{Irrep}(V'\otimes V^*)|} d_\mu \sum_{i=1}^{m_\mu}\sum_{k,l=1}^{m'_\nu} r^{(\mu,\nu)}_{\mathbb{S};i,i,k,l}\mathrm{Tr}_{\mathrm{in}}[T^{(\nu)\prime *}_{l,k}] = d_{\mathrm{out}}\mathbb{1}_{\mathrm{in}'}, \tag{2.94}$$

and if $(U',\mathcal{H}_{\mathrm{in}'})$ is irreducible too, then $\mathbb{S}$ is TP^2 *if and only if*

$$\sum_{\nu=1}^{|\mathrm{Irrep}(U'\otimes U^*)|} d'_\nu \sum_{\mu=1}^{|\mathrm{Irrep}(V'\otimes V^*)|} d_\mu \sum_{i=1}^{m_\mu}\sum_{k=1}^{m'_\nu} r^{(\mu,\nu)}_{\mathbb{S};i,i,k,k} = d_{\mathrm{out}}\cdot d_{\mathrm{in}'}, \tag{2.95}$$

where d'_ν is the dimension of the ν-th invariant subspace $\mathcal{H}^{(\nu)\prime}$ as defined in isotypic decomposition (2.81).

Proof Let $\mathbb{S}$ be a $\mathbf{G}$-covariant supermap on its output spaces in the form (2.83). Now, let us recall the TP^2 condition (2.29), which we may explicitly rewrite here as

$$\mathrm{Tr}_{\mathrm{out}'}[\mathscr{S}_{\mathbb{S}}(R)] = \mathbb{1}_{\mathrm{in}'} \quad \forall\, R\in\mathcal{B}(\mathcal{H}_{\mathrm{out}}\otimes\mathcal{H}_{\mathrm{in}}) \mid \mathrm{Tr}_{\mathrm{out}}[R]=\mathbb{1}_{\mathrm{in}}. \tag{2.96}$$

Then, we must consider

$$\begin{aligned}\mathrm{Tr}_{\mathrm{out}'}[\mathscr{S}_{\mathbb{S}}(R)] &= \sum_{\mu=1}^{|\mathrm{Irrep}(V'\otimes V^*)|}\sum_{i,j=1}^{m_\mu}\mathrm{Tr}_{\mathrm{out}',\mathrm{out},\mathrm{in}}[(\mathbb{1}_{\mathrm{out}',\mathrm{in}'}\otimes R^\top)(T^{(\mu)}_{j,i}\otimes R^{(\mu)}_{\mathbb{S};i,j}{}')] = \\ &= \sum_{\mu=1}^{|\mathrm{Irrep}(V'\otimes V^*)|}\sum_{i,j=1}^{m_\mu}\mathrm{Tr}_{\mathrm{out},\mathrm{in}}[(\mathbb{1}_{\mathrm{in}'}\otimes R^\top)(\mathrm{Tr}_{\mathrm{out}'}[T^{(\mu)}_{j,i}]\otimes R^{(\mu)}_{\mathbb{S};i,j}{}')].\end{aligned} \tag{2.97}$$

Since $T^{(\mu)}_{j,i}$ is an isometry mapping the irreducible module $\mathcal{H}^{(\mu)}_i$ into the equivalent module $\mathcal{H}^{(\mu)}_j$, then of course we have

$$\begin{aligned}T^{(\mu)}_{j,i} &= [V'_g\otimes V^*_g]|_{\mathcal{H}^{(\mu)}_j}T^{(\mu)}_{j,i}[V'_g\otimes V^*_g]|^\dagger_{\mathcal{H}^{(\mu)}_i} = \\ &= [V'_g\otimes V^*_g]T^{(\mu)}_{j,j}T^{(\mu)}_{j,i}T^{(\mu)\dagger}_{i,i}[V'_g\otimes V^*_g] = \\ &= [V'_g\otimes V^*_g]T^{(\mu)}_{j,i}[V'_g\otimes V^*_g]^\dagger \quad \forall\, g\in\mathbf{G},\end{aligned} \tag{2.98}$$

from which follows

$$\mathrm{Tr}_{\mathrm{out}'}[T^{(\mu)}_{j,i}] = V^*_g\mathrm{Tr}_{\mathrm{out}'}[T^{(\mu)}_{j,i}]V^\top_g \quad \forall\, g\in\mathbf{G}, \tag{2.99}$$

namely $\mathrm{Tr}_{\mathrm{out}'}[T^{(\mu)}_{j,i}]$ is an operator on $\mathcal{H}_{\mathrm{out}}$ which commutes with the action of $\mathbf{G}$. Then, thanks to the fact that $(V, \mathcal{H}_{\mathrm{out}})$ is irreducible by hypothesis, we are allowed to apply the Schur Lemma A.1, which yields

$$\mathrm{Tr}_{\mathrm{out}'}[T^{(\mu)}_{j,i}] = \lambda^{(\mu)}_{j,i} \mathbb{1}_{\mathrm{out}}, \tag{2.100}$$

with $\lambda^{(\mu)}_{j,i}$ determined by

$$\begin{aligned} \underset{\substack{\shortparallel \\ \delta_{ji}\, d_\mu}}{\mathrm{Tr}[T^{(\mu)}_{j,i}]} &= \lambda^{(\mu)}_{j,i} \cdot \mathrm{Tr}[\mathbb{1}_{\mathrm{out}}] = \lambda^{(\mu)}_{j,i} \cdot d_{\mathrm{out}}, \end{aligned} \tag{2.101}$$

i.e.

$$\lambda^{(\mu)}_{j,i} = \delta_{j,i} \frac{d_\mu}{d_{\mathrm{out}}}. \tag{2.102}$$

Then, substituting back in Eq. (2.97), we obtain

$$\begin{aligned} \mathrm{Tr}_{\mathrm{out}'}[\mathscr{S}_{\mathbb{S}}(R)] =& \sum_{\mu=1}^{|\mathrm{Irrep}(V'\otimes V^*)|} \sum_{i=1}^{m_\mu} \frac{d_\mu}{d_{\mathrm{out}}} \mathrm{Tr}_{\mathrm{out,in}}[(\mathbb{1}_{\mathrm{in}'} \otimes R^\top)(\mathbb{1}_{\mathrm{out}} \otimes R^{(\mu)}_{\mathbb{S};i,i}{}')] = \\ =& \sum_{\mu=1}^{|\mathrm{Irrep}(V'\otimes V^*)|} \sum_{i=1}^{m_\mu} \frac{d_\mu}{d_{\mathrm{out}}} \mathrm{Tr}_{\mathrm{in}}[(\mathbb{1}_{\mathrm{in}'} \otimes \mathrm{Tr}_{\mathrm{out}}[R^\top]) R^{(\mu)}_{\mathbb{S};i,i}{}'] = \\ =& \sum_{\mu=1}^{|\mathrm{Irrep}(V'\otimes V^*)|} \sum_{i=1}^{m_\mu} \frac{d_\mu}{d_{\mathrm{out}}} \mathrm{Tr}_{\mathrm{in}}[R^{(\mu)}_{\mathbb{S};i,i}{}'] \quad \forall\, R \in \mathcal{B}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}) \mid \mathrm{Tr}_{\mathrm{out}}[R] = \mathbb{1}_{\mathrm{in}}. \end{aligned} \tag{2.103}$$

Thus, we have proved condition (2.93).

To obtain condition (2.94), it is sufficient to make the substitution

$$R^{(\mu)}_{\mathbb{S};i,i}{}' \mapsto \sum_{\nu=1}^{|\mathrm{Irrep}(U'\otimes U^*)|} \sum_{k,l=1}^{m'_\nu} r^{(\mu,\nu)}_{\mathbb{S};i,i,k,l} T^{(\nu)\prime *}_{l,k}. \tag{2.104}$$

in Eq. (2.93): notice that the above substitution is the one which allows one to switch from Eq. (2.83) to Eq. (2.84).

Finally, just as we found that $\mathrm{Tr}_{\mathrm{out}'}[T^{(\mu)}_{j,i}]$ commuted with the action of $\mathbf{G}$, it is easy to realize that $\mathrm{Tr}_{\mathrm{in}}[T^{(\nu)\prime *}_{l,k}]$ is an operator on $\mathcal{H}_{\mathrm{in}'}$ which commutes with the action U' of $\mathbf{G}$. Then, thanks to the fact that $(U', \mathcal{H}_{\mathrm{in}'})$ is irreducible by hypothesis, once again we are allowed to apply the Schur Lemma A.1, which yields

$$\mathrm{Tr}_{\mathrm{in}}[T^{(\nu)\prime *}_{l,k}] = \delta_{lk} \frac{d'_\nu}{d_{\mathrm{in}'}} \mathbb{1}_{\mathrm{in}'}. \tag{2.105}$$

Substituting in condition (2.94) yields condition (2.95). ■

Remark 2.11 In the previous proof, the key hypothesis which is needed in order to implement the normalization of the input channel is the irreducibility of $(V, \mathcal{H}_{\text{out}})$, namely it was this hypothesis that let us take the partial trace of $R^\top$ on $\mathcal{H}_{\text{out}}$ separately in Eq. (2.97). On the contrary, irreducibility of $(U', \mathcal{H}_{\text{in}'})$ is only required if one wants to further refine condition (2.93). ▲

2.4 Quantum Superchannels

So far, we have obtained full characterizations of C^2P^2 and TP^2 supermaps in terms of their representing maps: this allows us to study the relation between C^2P^2&TP^2 supermaps and physical transformations of Quantum Maps.

2.4.1 A Stinespring Theorem for Supermaps

Up to this point our treatment has been highly speculative, as our axiomatic approach could not guarantee all Quantum Supermaps to be physically realizable by means of some quantum circuit.

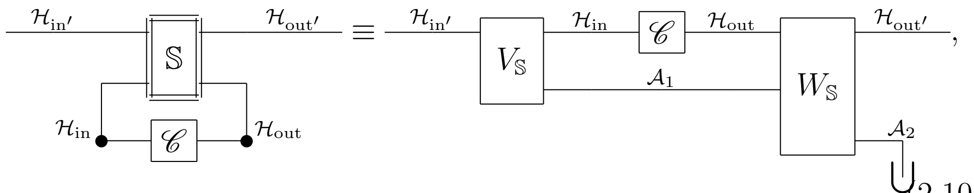
Notice that we find ourselves in the very same situation of the beginning of Section 1.3: there, we were eventually able to prove that all Quantum Maps were physical (thanks to Stinespring Theorem 1.7). Here, we need an an analogous result to the one of Stinespring for the case of supermaps.

More precisely, in Section 1.3 we proved that all Quantum Maps (satisfying Axioms 1.1, 1.2 and 1.4) could be regarded as state evolutions of open systems and, conversely, that all of such evoltions could be described in terms of Quantum Maps. Analogously, in the present Subsection we will prove that all Quantum Supermaps (satisfying Proposition 2.4 and Axiom 2.5) can be regarded as some quantum circuital scheme of which the input map is a composing gate and, conversely, that all of such schemes can be described in terms of Quantum Supermaps.

This remarkable result is provided by the following Theorem.

Theorem 2.14 (A Stinespring Theorem for Supermaps) *Let $\mathbb{S}$ be a su-*

permap. Then, $\mathbb{S}$ *is* $\mathrm{C^2P^2\&TP^2}$ *if and only if*



(2.106)

where $V_{\mathbb{S}}$ *and* $W_{\mathbb{S}}$ *are proper isometric operators, and ancillary Hilbert spaces* $\mathcal{A}_1$*,* $\mathcal{A}_2$ *were introduced.*

Proof First, let us prove the 'only if' part, namely the fact that all $\mathrm{C^2P^2\&TP^2}$ supermaps may be physically implemented using the above scheme.

So, let $\mathbb{S}$ be a $\mathrm{C^2P^2\&TP^2}$ supermap: then, thanks to Corollary 2.7 we can find a Quantum Channel[3] $\mathscr{E}_{\mathbb{S}}^{\top} : \mathcal{B}(\mathcal{H}_{\mathrm{in}'}) \to \mathcal{B}(\mathcal{H}_{\mathrm{in}})$ satisfying Eq. (2.45) for all $A' \in \mathcal{T}(\mathcal{H}_{\mathrm{in}'})$. If we denote with $\{K_z \mid z \in Z\}$ and $\{E_a \mid a \in A\}$ two canonical Kraus decompositions for $\mathscr{S}_{\mathbb{S}}$ and $\mathscr{E}_{\mathbb{S}}$, respectively, condition (2.45) may be rewritten as

$$\begin{array}{ccc}
\sum_{z\in Z} K_z^{\dagger}(\mathbb{1}_{\mathrm{out}'} \otimes A')K_z & = & \mathbb{1}_{\mathrm{out}} \otimes \sum_{a\in A} E_a^{\dagger} A' E_a \\
\shortparallel & & \shortparallel \\
\sum_{z\in Z}\sum_{n=1}^{d_{\mathrm{out}'}} K_z^{\dagger}\big[|n\rangle \otimes \mathbb{1}\big]A'\big[\langle n| \otimes \mathbb{1}\big]K_z & & \sum_{a\in A}\sum_{m=1}^{d_{\mathrm{out}}} \big[|m\rangle \otimes E_a^{\dagger}\big]A'\big[\langle m| \otimes E_a\big]
\end{array} \tag{2.107}$$

for all $A' \in \mathcal{T}(\mathcal{H}_{\mathrm{in}'})$. This shows that $\{\tilde{K}_n^{(z)\dagger} \mid z \in Z, n = 1, \ldots, d_{\mathrm{out}'}\}$ and $\{\tilde{E}_m^{(a)\dagger} \mid a \in A, m = 1, \ldots, d_{\mathrm{out}}\}$ are two Kraus decompositions for the same map $A' \mapsto \mathbb{1}_{\mathrm{out}} \otimes \mathscr{E}_{\mathbb{S}}^{\top}(A')$, provided that we have put

$$\begin{aligned}
\tilde{K}_n^{(z)} &\doteq \big[{}_{\mathrm{out}'}\langle n| \otimes \mathbb{1}_{\mathrm{in}'}\big]K_z, \\
\tilde{E}_m^{(a)} &\doteq {}_{\mathrm{out}}\langle m| \otimes E_a,
\end{aligned} \tag{2.108}$$

which may be reversed as

$$\begin{aligned}
K_z &= \sum_{n=1}^{d_{\mathrm{out}'}} \big[|n\rangle_{\mathrm{out}'} \otimes \mathbb{1}_{\mathrm{in}'}\big]\tilde{K}_n^{(z)}, \\
E_a &= \frac{1}{d_{\mathrm{out}}}\mathrm{Tr}_{\mathrm{out}}\left[\sum_{m=1}^{d_{\mathrm{out}}} |m\rangle_{\mathrm{out}} \otimes \tilde{E}_m^{(a)}\right].
\end{aligned} \tag{2.109}$$

[3] See Remarks 2.5 and 2.6.

Furthermore, since operators $\{E_a\}$ are orthogonal, then we see that operators $\{\tilde{E}_m^{(a)}\}$ are orthogonal as well, so that they form a *canonical* Kraus decomposition. Then, thanks to Theorem 1.4, there must exist a $([Z]\cdot d_{\mathrm{out}'})\times([A]\cdot d_{\mathrm{out}})$ isometric matrix U such that

$$\tilde{K}_n^{(z)} = \sum_{(a,m)\in A\times\{1,\dots,d_{\mathrm{out}}\}} U_{(z,n),(a,m)}\tilde{E}_m^{(a)} \tag{2.110}$$

for all $(z,n)\in Z\times\{1,\dots,d_{\mathrm{out}'}\}$. So we obtain

$$\begin{aligned} K_z &= \sum_{n=1}^{d_{\mathrm{out}'}} \left[|n\rangle_{\mathrm{out}'}\otimes \mathbb{1}_{\mathrm{in}'}\right]\tilde{K}_n^{(z)} = \\ &= \sum_{a\in A}\sum_{m=1}^{d_{\mathrm{out}}}\sum_{n=1}^{d_{\mathrm{out}'}} U_{(z,n),(a,m)}\left[|n\rangle_{\mathrm{out}'}\otimes \mathbb{1}_{\mathrm{in}'}\right]\tilde{E}_m^{(a)} = \\ &= \sum_{a\in A}\left[\sum_{m=1}^{d_{\mathrm{out}}}\sum_{n=1}^{d_{\mathrm{out}'}} U_{(z,n),(a,m)}|n\rangle_{\mathrm{out}'\ \mathrm{out}}\langle m|\right]\otimes E_a = \\ &= \sum_{a\in A} F_a^{(z)}\otimes E_a \end{aligned} \tag{2.111}$$

where we have put

$$F_a^{(z)} \doteq \sum_{m=1}^{d_{\mathrm{out}}}\sum_{n=1}^{d_{\mathrm{out}'}} U_{(z,n),(a,m)}|n\rangle_{\mathrm{out}'\ \mathrm{out}}\langle m|. \tag{2.112}$$

Eq. (2.111) has the important consequence that, if we feed the supermap $\mathbb{S}$ with a Quantum Channel $\mathscr{C}$, then Choi operator of the output Quantum Channel may be written as

$$\begin{aligned} R_{\mathbb{S}(\mathscr{C})} &= \mathscr{S}_{\mathbb{S}}(R_{\mathscr{C}}) = \\ &= \sum_{z\in Z}\sum_{x\in X} K_z|M_x\rangle\rangle_{\mathrm{out,in}}\langle\langle M_x|K_z^\dagger = \\ &= \sum_{z\in Z}\sum_{x\in X} |\textstyle\sum_{a\in A} F_a^{(z)}M_xE_a^\top\rangle\rangle_{\mathrm{out}',\mathrm{in}'}\langle\langle\sum_{a'\in A} F_{a'}^{(z)}M_xE_{a'}^\top|, \end{aligned} \tag{2.113}$$

having introduced the canonical Kraus decomposition $\{M_x \mid x\in X\}$ for the input Quantum Channel $\mathscr{C}$. This shows that the output Quantum Channel $\mathbb{S}(\mathscr{C})$ admits a Kraus decomposition $\{M'_{(x,z)} \mid (x,z)\in X\times Z\}$ explicitly given by

$$M'_{(x,z)} = \sum_{a\in A} F_a^{(z)}M_xE_a^\top, \tag{2.114}$$

so that

$$[\mathbb{S}(\mathscr{C})](A') = \sum_{x\in X}\sum_{z\in Z}\sum_{a,a'\in A} F_a^{(z)} M_x E_a^\top A' E_{a'}^* M_x^\dagger F_{a'}^{(z)\dagger} \quad \forall\, A' \in \mathcal{T}(\mathcal{H}_{\mathrm{in}'}). \tag{2.115}$$

Now, let us introduce two ancillary Hilbert spaces, $\mathcal{A}_1 \cong \mathbb{C}^{|A|}$ and $\mathcal{A}_2 \cong \mathbb{C}^{|Z|}$, and two linear operators V, W respectively defined by

$$V \doteq \sum_{a\in A} E_a^\top \otimes |a\rangle_{\mathcal{A}_1} : \mathcal{H}_{\mathrm{in}'} \to \mathcal{H}_{\mathrm{in}} \otimes \mathcal{A}_1, \tag{2.116}$$

$$W \doteq \sum_{z\in Z}\sum_{a\in A} F_a^{(z)} \otimes |z\rangle_{\mathcal{A}_2\ \mathcal{A}_1}\langle a| : \mathcal{H}_{\mathrm{out}} \otimes \mathcal{A}_1 \to \mathcal{H}_{\mathrm{out}'} \otimes \mathcal{A}_2. \tag{2.117}$$

Then we have

$$\begin{aligned} V^\dagger V &= \sum_{a,a'\in A} \left[E_{a'}^* \otimes \langle a'|\right]\left[E_a^\top \otimes |a\rangle\right] = \\ &= \sum_{a\in A} E_a^* E_a^\top = \\ &= \mathbb{1}_{\mathrm{in}'}, \end{aligned} \tag{2.118}$$

namely V is an isometry thanks to the fact that $\mathscr{E}_{\mathbb{S}}^\top$ is TP (i.e. that $\mathbb{S}$ is $\mathrm{T}^2\mathrm{P}^2$). Furthermore,

$$\begin{aligned} W^\dagger W &= \sum_{z,z'\in Z}\sum_{a,a'\in A} \left[F_{a'}^{(z')\dagger} \otimes |a'\rangle\langle z'|\right]\left[F_a^{(z)} \otimes |z\rangle\langle a|\right] = \\ &= \sum_{z\in Z}\sum_{a,a'\in A} F_{a'}^{(z)\dagger} F_a^{(z)} \otimes |a'\rangle\langle a| = \\ &= \sum_{z\in Z}\sum_{a,a'\in A}\sum_{m,m'=1}^{d_{\mathrm{out}}}\sum_{n,n'=1}^{d_{\mathrm{out}'}} U^*_{(z,n'),(a',m')} |m'\rangle\langle n'| U_{(z,n),(a,m)} |n\rangle\langle m| \otimes |a'\rangle\langle a| = \\ &= \sum_{a,a'\in A}\sum_{m,m'=1}^{d_{\mathrm{out}}} (U^\dagger U)_{(a',m'),(a,m)} |m'\rangle\langle m| \otimes |a'\rangle\langle a| = \\ &= \mathbb{1}_{\mathrm{out}} \otimes \mathbb{1}_{\mathcal{A}_1}, \end{aligned} \tag{2.119}$$

thanks to the fact that U is an isometric matrix: this proves that W is an isometry as well.

This part of the proof is concluded by considering

$$
\begin{aligned}
&\mathrm{Tr}_{\mathcal{A}_2}\left[W\big([\mathscr{C}\otimes \mathbb{1}_{\mathcal{T}(\mathcal{A}_1)}](VA'V^\dagger)\big)W^\dagger\right] = \\
&= \sum_{a,a'\in A} \mathrm{Tr}_{\mathcal{A}_2}\left[W\Big([\mathscr{C}\otimes \mathbb{1}_{\mathcal{T}(\mathcal{A}_1)}]\Big(E_a^\top A' E_{a'}^* \otimes |a\rangle\langle a'|\Big)\Big)W^\dagger\right] = \\
&= \sum_{x\in X}\sum_{a,a'\in A} \mathrm{Tr}_{\mathcal{A}_2}\left[W\Big(M_x E_a^\top A' E_{a'}^* M_x^\dagger \otimes |a\rangle\langle a'|\Big)W^\dagger\right] = \\
&= \sum_{z,z'\in Z}\sum_{\tilde{a},\tilde{a}'\in A}\sum_{x\in X}\sum_{a,a'\in A} \mathrm{Tr}_{\mathcal{A}_2}\Bigg[(F_{\tilde{a}}^{(z)} \otimes |z\rangle\langle \tilde{a}|) \\
&\qquad\qquad \Big(M_x E_a^\top A' E_{a'}^* M_x^\dagger \otimes |a\rangle\langle a'|\Big)(F_{\tilde{a}'}^{(z')\dagger} \otimes |\tilde{a}'\rangle\langle z'|)\Bigg] = \\
&= \sum_{z,z'\in Z}\sum_{x\in X}\sum_{a,a'\in A} \mathrm{Tr}_{\mathcal{A}_2}\left[\Big(F_a^{(z)} M_x E_a^\top A' E_{a'}^* M_x^\dagger F_{a'}^{(z')\dagger} \otimes |z\rangle\langle z'|\Big)\right] = \\
&= \sum_{z\in Z}\sum_{x\in X}\sum_{a,a'\in A} F_a^{(z)} M_x E_a^\top A' E_{a'}^* M_x^\dagger F_{a'}^{(z)\dagger},
\end{aligned}
\tag{2.120}
$$

which evidently coincides with $[\mathbb{S}(\mathscr{C})](A')$, thanks to Eq. (2.115)

Finally we have to prove that, for all isometric operators $V : \mathcal{H}_{\mathrm{in}'} \to \mathcal{H}_{\mathrm{in}} \otimes \mathcal{A}_1$ and $W : \mathcal{H}_{\mathrm{out}} \otimes \mathcal{A}_1 \to \mathcal{H}_{\mathrm{out}'} \otimes \mathcal{A}_2$, the supermap $\tilde{\mathbb{S}}$ such that

$$
\tilde{\mathbb{S}}(\mathscr{C}) = \mathrm{Tr}_{\mathcal{A}_2}\left[W\Big([\mathscr{C}\otimes \mathbb{1}_{\mathcal{T}(\mathcal{A}_1)}]\big(V \bullet V^\dagger\big)\Big)W^\dagger\right] \tag{2.121}
$$

is $\mathrm{C}^2\mathrm{P}^2$&TP^2. Clearly, $\tilde{\mathbb{S}}$ is QCP: indeed, scheme (2.106) is a Quantum Channel if and only if the input map $\mathscr{C}$ is a Quantum Channel. Then, thanks to Lemma 2.9, $\tilde{\mathbb{S}}$ is TP^2 as well. Finally, to check that it is $\mathrm{C}^2\mathrm{P}^2$, consider Eq. (2.120), where now $\{M_x\}$ is a canonical Kraus decomposition of a CP map $\mathscr{C}$ – which is not necessarily TP. Then, recalling Definition 2.6 of Choi operators for supermaps, and Remark 2.2, the last line tells us that

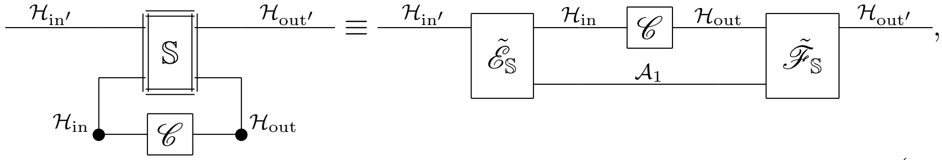
$$
R_{\tilde{\mathbb{S}}} = \sum_{z\in Z} \big|\, \textstyle\sum_{a\in A} F_a^{(z)} \otimes E_a \big\rangle\!\big\rangle\big\langle\!\big\langle\, \textstyle\sum_{a'\in A} F_{a'}^{(z)} \otimes E_{a'} \big|, \tag{2.122}
$$

which proves that $\tilde{\mathbb{S}}$ is $\mathrm{C}^2\mathrm{P}^2$ thanks to Corollary 2.5. ■

Remark 2.12 The proof of Theorem 2.14 evidently suggests an explicit way for implementing any $\mathrm{C}^2\mathrm{P}^2$&TP^2 supermap $\mathbb{S}$. It should also be evident that there exist infinite physical implementation of the same supermap: in fact, the purpose of the above Theorem was not that of finding the optimal implementation (whatever criterion of optimality one chooses), but just to show that a physical implementation of all $\mathrm{C}^2\mathrm{P}^2\&\mathrm{TP}^2$ supermaps is possible.

▲

Remark 2.13 Consider the proof of Theorem 2.14: notice that the second isometry, $W_{\mathbb{S}}$, might be as well replaced by a Quantum Channel $\tilde{\mathscr{F}}_{\mathbb{S}} \doteq \mathrm{Tr}_{\mathcal{A}_2}[W_{\mathbb{S}} \bullet W_{\mathbb{S}}^\dagger] : \mathcal{T}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{A}_1) \to \mathcal{T}(\mathcal{H}_{\mathrm{out}'})$. Furthermore any isometry $V_{\mathbb{S}}$ is, trivially, a Quantum Channel, so that we may conclude that all $\mathrm{C}^2\mathrm{P}^2\&\mathrm{TP}^2$ supermaps are implemented by



(2.123)

where $\tilde{\mathscr{E}}_{\mathbb{S}}$ and $\mathscr{F}_{\mathbb{S}}$ are Quantum Channels depending on $\mathbb{S}$ — the $\tilde{}$ was necessary in order to distinguish the first Quantum Channel from the map $\mathscr{E}_{\mathbb{S}}$ introduced in Theorem 2.6. Conversely, it is easy to prove that all supermaps satisfying Eq. (2.123) are $\mathrm{C}^2\mathrm{P}^2\&\mathrm{TP}^2$ as well: so, we conclude this Remark with the following

Corollary 2.15 (to Theorem 2.14) *Let $\mathbb{S}$ be a linear map. Then, it is* $\mathrm{C}^2\mathrm{P}^2\&\mathrm{TP}^2$ *if and only if it may be physically implemented as in Eq. (2.123).*

▲

The quantum circuit on the right-hand side of Eq. (2.123) evidently represents the most general quantum circuit which implements $\mathscr{C}$ as a composing gate. Since we have shown that each and every so-conceived quantum circuit is represented by a $\mathrm{C}^2\mathrm{P}^2\&\mathrm{TP}^2$ supermap, it is natural to state the following

Definition 2.8 (Quantum Superchannel) *Let $\mathbb{S}$ be a linear supermap: we will say that it is a* Quantum Superchannel *when it is* $\mathrm{C}^2\mathrm{P}^2\&\mathrm{TP}^2$. *Furthermore, we will denote with* $\mathrm{QSC}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}; \mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'})$ *the set of Quantum Superchannels of* $\mathcal{L}(\mathcal{T}(\mathcal{H}_{\mathrm{in}}), \mathcal{T}(\mathcal{H}_{\mathrm{out}}))$ *into* $\mathcal{L}(\mathcal{T}(\mathcal{H}_{\mathrm{in}'}), \mathcal{T}(\mathcal{H}_{\mathrm{out}'}))$.

Notice that Definition 2.8 of Quantum Superchannels is strictly analogous to Definition 1.7 of Quantum Channels.

Furthermore, it is evident that the set $\mathrm{QSC}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}; \mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'})$ of Quantum Superchannels is a convex set: indeed, thanks to Corollary 2.5 and Remark 2.7 we have

$$\begin{aligned}\mathrm{QSC}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}; \mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'}) &= [\mathrm{C}^2\mathrm{P}^2 \cap \mathrm{TP}^2](\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}; \mathcal{H}_{\mathrm{in}'}, \mathcal{H}_{\mathrm{out}'}) \cong \\ &\cong [\Omega \cap \Theta](\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'} \otimes \mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})\end{aligned} \tag{2.124}$$

where Ω is the convex cone of positive Choi operators and Θ is the affine space of normalized Choi operators.

Remark 2.14 Notice that Quantum Superchannels generalize Quantum Channels. Indeed, consider a linear supermap $\mathbb{S}$ between $\mathbb{C}$ and $\mathcal{L}(\mathcal{T}(\mathcal{H}_{\mathrm{in}}), \mathcal{T}(\mathcal{H}_{\mathrm{out}}))$. Then, $\mathbb{S}$ it is TP^2 if and only if there exists a unital map $\mathscr{E}_{\mathbb{S}} : \mathbb{C} \to \mathcal{T}(\mathcal{H}_{\mathrm{in}})$ such that $\mathrm{Tr}_{\mathrm{out}}[\mathscr{S}_{\mathbb{S}}(r)] = \mathscr{E}_{\mathbb{S}}(r)$ for all $r \in \mathbb{C}$, namely iff

$$\mathrm{Tr}_{\mathrm{out}}[\mathscr{S}_{\mathbb{S}}(1)] = \mathbb{1}_{\mathrm{in}}. \tag{2.125}$$

However, its Choi operator is easily checked to be $R_{\mathbb{S}} = \mathscr{S}_{\mathbb{S}}(1) \in \mathcal{B}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$, so that $\mathbb{S}$ is a Quantum Superchannel if and only if the map of $\mathcal{H}_{\mathrm{in}}$ into $\mathcal{H}_{\mathrm{out}}$ corresponding to the Choi operator $R_{\mathbb{S}}$ is a Quantum Channel:

$$\mathrm{QC}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}) \cong \mathrm{QSC}(\mathbb{C}, \mathbb{C}; \mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}). \tag{2.126}$$

▲

Chapter 3

Cloning of Unitaries

In the present Chapter, the problem of cloning groups of state transformations is introduced as an application of the formalism that was developed in Chapter 2. In Section 3.1 the general case in which the group of unitaries to be cloned is any compact group is investigated: since an ideal cloning is proved to be impossible in the general case, a strategy for the search of an optimal cloner is outlined. In Section 3.2, the particular case of universal cloning (namely, the problem of cloning all unitary state transformations) is solved for qu*d*its using the strategy and the main results that were developed in the preceding Section.

3.1 Introduction to the general case

In Chapter 2, the mathematical formalism of Quantum Supermaps was introduced in order to study the most general physical transformations of Quantum Maps: in Section 2.4 it was further proved that, under the axiomatization provided in Section 2.1, all Quantum Superchannels may be physically implemented using quantum circuits consisting of certain Quantum Channels. Then, a natural question is whether there exist significative physical situations where the implementation of some specific Quantum Supermap has some practical application.

3.1.1 On the Impossibility of Ideal Cloning

In particular, consider the case in which *one* quantum system undergoes some state transformation which is due to an unknown (or a partially known) interaction with some physical device: then we may wonder whether there exists a quantum circuit, of which such a physical device is part, that is equivalent to performing the same state transformation on *two* quantum systems separately. Of course, we may refer to this situation as a 'cloning of Quantum Maps'. Indeed, in the supermap formalism, the problem may be restated as follows: For any unknown (or partially known) input map $\mathscr{C} : \mathcal{T}(\mathcal{H}_{\mathrm{in}}) \to \mathcal{T}(\mathcal{H}_{\mathrm{out}})$, find a Quantum Supermap $\mathbb{S} : \mathcal{L}(\mathcal{T}(\mathcal{H}_{\mathrm{in}}), \mathcal{T}(\mathcal{H}_{\mathrm{out}})) \to \mathcal{L}(\mathcal{T}(\mathcal{H}_{\mathrm{in}_1} \otimes \mathcal{H}_{\mathrm{in}_2}), \mathcal{T}(\mathcal{H}_{\mathrm{out}_1} \otimes \mathcal{H}_{\mathrm{out}_2}))$ which maps $\mathscr{C}$ into $\mathbb{S}(\mathscr{C}) = \mathscr{C}^{\otimes 2}$.

Whilst the cloning of Quantum Maps has received very little attention in literature so far, interest in the problem of cloning Quantum States arised quite early in the historical development of Quantum Information and Quantum Computation, due to the importance of the process of copying information. Indeed, in 1982, Wootters and Zurek proved the well-known *no-cloning theorem* [10], in which it is shown that there cannot exist quantum devices cloning perfectly all pure states in a finite Hilbert space. The original proof may be given by contradiction: indeed, let $\mathscr{C}_{\mathrm{id}}$ be an ideal cloning linear map, namely let

$$\mathscr{C}_{\mathrm{id}}(\rho) = \rho^{\otimes 2} \quad \forall\, \rho \in \mathrm{S}(\mathcal{H}). \tag{3.1}$$

Then, using the convexity of $\mathrm{S}(\mathcal{H})$ and the linearity of $\mathscr{C}_{\mathrm{id}}$, for any ρ_0, ρ_1 with orthogonal support we obtain the contradiction

$$\begin{array}{ccl} \mathscr{C}_{\mathrm{id}}(\rho_p) & = & p\rho_1^{\otimes 2} + (1-p)\rho_0^{\otimes 2} \\ \shortparallel & & \nparallel \\ \rho_p^{\otimes 2} & = & p^2\rho_1^{\otimes 2} + (1-p)^2\rho_0^{\otimes 2} + p(1-p)[\rho_0 \otimes \rho_1 + \rho_1 \otimes \rho_0], \end{array} \tag{3.2}$$

where we have defined $\rho_p \doteq p\rho_1 + (1-p)\rho_0$, $p \in (0,1)$.

So, we expect a similar result for the cloning of Quantum Maps: indeed, this result is almost identical.

Theorem 3.1 (No-go Theorem for the Cloning of Quantum Maps) *The ideal cloning of Quantum Channels is not achievable by means of any linear supermap.*

Proof Let us give the proof by contradiction: so, suppose that $\mathbb{S}_{\mathrm{id}}$ is a linear supermap such that $\mathbb{S}_{\mathrm{id}}(\mathscr{C}) = \mathscr{C}^{\otimes 2}$ for all Quantum Channels $\mathscr{C}$; or, equivalently, let its representing map $\mathscr{S}_{\mathbb{S}_{\mathrm{id}}}$ be such that

$$\mathscr{S}_{\mathbb{S}_{\mathrm{id}}}(R) = R^{\otimes 2} \quad \forall\, R \in [\Omega \cap \mathrm{N}_{\mathbb{1}_{\mathrm{in}}}](\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}). \tag{3.3}$$

Now, since $[\Omega \cap \mathrm{N}_{\mathbb{1}_{\mathrm{in}}}](\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$ is convex, then for every two operators R_0 and R_1 in this space with orthogonal supports we have

$$R_p \doteq pR_1 + (1-p)R_0 \ \in [\Omega \cap \mathrm{N}_{\mathbb{1}_{\mathrm{in}}}](\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}) \quad \forall\, p \in [0,1]. \tag{3.4}$$

Thus, just as in the case of Quantum States, we obtain the contradiction

$$\begin{array}{rcl} \mathscr{S}_{\mathbb{S}_{\mathrm{id}}}(R_p) & = & pR_1^{\otimes 2} + (1-p)R_0^{\otimes 2} \\ \shortparallel & & \nparallel \\ R_p^{\otimes 2} & = & p^2 R_1^{\otimes 2} + (1-p)^2 R_0^{\otimes 2} + p(1-p)[R_0 \otimes R_1 + R_1 \otimes R_0], \end{array} \tag{3.5}$$

for all $p \in (0,1)$, that proves the Theorem. ■

In the case of quantum states, actually, the perfect cloning of orthogonal sets of states is achievable [11], thus expliciting the fact that classical information can by perfectly copied, at least in principle. For instance, if one defines the linear map $\mathscr{C}$ by

$$\mathscr{C}(|i\rangle\langle j|) \doteq \delta_{i,j} |i\rangle\langle i| \otimes |i\rangle\langle i|, \tag{3.6}$$

then $\mathscr{C}$ is a Quantum Channel which is able to perfectly clone all (and only) pure states in the orthonormal basis $\{|i\rangle \mid i = 1, \ldots, d_{\mathcal{H}}\}$ of $\mathcal{H}$. Similar examples may be found for the cloning of Quantum Maps: for instance, one may define the linear supermap $\mathbb{S}$ by

$$\mathscr{S}_{\mathbb{S}}(R) \doteq \frac{1}{d_{\mathrm{in}}} \sum_{i=1}^{d_{\mathrm{in}}} \langle i|\mathrm{Tr}_{\mathrm{in}}[R]|i\rangle \; |i\rangle_{\mathrm{out}_1}\langle i| \otimes |i\rangle_{\mathrm{out}_2}\langle i| \otimes \mathbb{1}_{\mathrm{in}_1} \otimes \mathbb{1}_{\mathrm{in}_2}, \tag{3.7}$$

so that $\mathbb{S}$ is easily checked to be a Quantum Superchannel which is able to perfectly clone all Quantum Channels $\{\mathscr{C}^{(k)} \mid k = 1, \ldots, d_{\mathrm{out}}\}$ with $\mathscr{C}^{(k)}(A) = \mathrm{Tr}[A]|k\rangle\langle k|$.

Another interesting possibilit in the case of quantum states is given by producing approximate copies of the input state to be cloned [12]: in the following, we will adopt such a strategy for the cloning of maps.

3.1.2 Cloning of Unitary Transformations

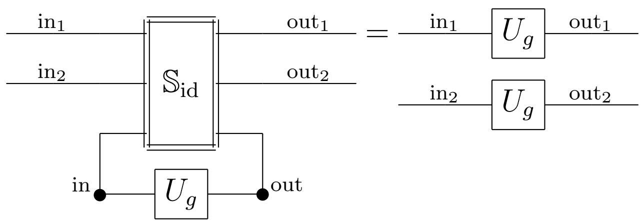
In Section 1.1 it was shown that all unitary transformations, when seen from a local point of view, may be represented by Quantum Channels, and in Section 1.3 it was proved that, conversely, all Quantum Channels may be regarded as some local part of a larger unitary transformation: then we realize that the problem of cloning Quantum Channels is perfectly equivalent to that of cloning unitary transformations.

Clearly, then, Theorem 3.1 implies that there must exist no linear supermap which is able to clone perfectly all unitary transformations on a given

Hilbert space[1]. On the other hand, it might still be possible to clone a certain subset of unitaries perfectly, or else to clone all unitaries approximately.

However, we shall consider a set of input maps to be cloned as the set $U_{\mathbf{G}} = \{U_g \mid g \in \mathbf{G}\}$ of unitaries which forms (the unitary representation of) a compact group $\mathbf{G}$; this particular choice is due to the fact that physical transformations often correspond to elements in a fixed group: for instance, the phase shift of a laser beam may be parametrized by elements in the phase group $\mathbf{U}(1)$.

So, let $(U, \mathcal{H})$ be a unitary representation of a group $\mathbf{G}$ on the Hilbert space $\mathcal{H} \cong \mathbb{C}^d$. Then, an ideal 1-to-2 Unitary Cloning Quantum Superchannel $\mathbb{S}_{\text{id}}$ must take unitary transformations $U_g \bullet U_g^\dagger$ as input and return the two unitaries $U_g^{\otimes 2} \bullet U_g^{\dagger \otimes 2}$ as output, for every g in $\mathbf{G}$. This might be sketched as follows:



$\forall\, g \in \mathbf{G}$, (3.8)

where all involved Hilbert spaces are isomorphic to the same d-dimensional Hilbert space $\mathcal{H}$. We remark the fact that in Theorem 3.1 it was proved the impossibility of cloning the *whole* set of unitary transformations on $\mathcal{H}$ via a linear supermap so that, depending on the choice of $\mathbf{G}$, such a $\mathbb{S}_{\text{id}}$ may or may not exist, actually. On the other hand, we expect $\mathbb{S}_{\text{id}}$ to exist only in trivial cases, so that in the following we will suppose that it does not.

Then, we will seek the optimal fixed Quantum Superchannel $\overline{\mathbb{S}}$ performing such a cloning of unitaries in an approximate way. Of course, we must define explicitly the notion of 'optimality': in the following we will be using the following

Definition 3.1 (Optimal Cloner) *Let $\overline{\mathbb{S}}$ be a Quantum Superchannel, and let $\mu(\bullet, \bullet)$ be a distance measure on the set of its output maps. Then, we will say that $\overline{\mathbb{S}}$ is an* optimal 1-to-2 unitary cloner *when the distance of its output map from that of the ideal cloner $\mathbb{S}_{\text{id}}$, averaged on the whole set $U_{\mathbf{G}}$ of unitaries to be cloned, is minimal.*

Thanks to Choi isomorphism (1.28), it is a natural choice to take[2]

$$\mu(\mathscr{C}', \mathscr{E}') \doteq (R_{\mathscr{C}'}, R_{\mathscr{E}'}), \tag{3.9}$$

[1]Indeed, if one were able to clone all unitaries, then it would be possible to deny Theorem 3.1 just by considering the unitary transformations locally.

[2] Note that if $\mathscr{C}'$ and $\mathscr{E}'$ are Quantum Channels, then $\mu(\mathscr{C}', \mathscr{E}') \in \mathbb{R}^+$, thanks to Theorem 1.3.

where $(\bullet, \bullet)$ is the usual Hilbert-Schmidt scalar product in $\mathcal{B}(\mathcal{H}_{\mathrm{out}'} \otimes \mathcal{H}_{\mathrm{in}'})$. Then, for every Quantum Superchannel $\mathbb{S}$ we have

$$\begin{aligned} \mu\big(\mathbb{S}(U_g \bullet U_g^\dagger), \mathbb{S}_{\mathrm{id}}(U_g \bullet U_g^\dagger)\big) &= \mu\big(\mathbb{S}(U_g \bullet U_g^\dagger), U_g^{\otimes 2} \bullet U_g^{\dagger\otimes 2}\big) = \\ &= \big(R_{\mathbb{S}(U_g \bullet U_g^\dagger)}, R_{U_g^{\otimes 2} \bullet U_g^{\dagger\otimes 2}}\big). \end{aligned} \tag{3.10}$$

Using Eq. (1.28) we obtain

$$\begin{aligned} R_{U_g \bullet U_g^\dagger} &= (U_g \otimes \mathbb{1})|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|(U_g^\dagger \otimes \mathbb{1}) = \\ &= |U_g\rangle\rangle\langle\langle U_g| \end{aligned} \tag{3.11}$$

so that we have trivially $R_{U_g^{\otimes 2} \bullet U_g^{\dagger\otimes 2}} = |U_g\rangle\rangle\langle\langle U_g|^{\otimes 2}$ and, by the inverse isomorphism formula (1.30),

$$\begin{aligned} R_{\mathbb{S}(U_g \bullet U_g^\dagger)} &= \mathscr{S}_{\mathbb{S}}(R_{U_g \bullet U_g^\dagger}) = \\ &= \mathrm{Tr}_{\mathrm{out,in}}\left[\left(\mathbb{1}_{\mathrm{out}_1} \otimes \mathbb{1}_{\mathrm{out}_2} \otimes R_{U_g \bullet U_g^\dagger}^\top\right) R_{\mathbb{S}}\right] = \\ &= \Big[\mathbb{1}_{\mathrm{out}_1} \otimes \mathbb{1}_{\mathrm{out}_2} \otimes {}_{\mathrm{out,in}}\langle\langle U_g^*|\Big] R_{\mathbb{S}} \Big[\mathbb{1}_{\mathrm{out}_1} \otimes \mathbb{1}_{\mathrm{out}_2} \otimes |U_g^*\rangle\rangle_{\mathrm{out,in}}\Big], \end{aligned} \tag{3.12}$$

thanks to the fact that $|U_g\rangle\rangle\langle\langle U_g|^\top = |U_g^*\rangle\rangle\langle\langle U_g^*|$. Finally, substituting in Eq. (3.10), we obtain

$$\mu\big(\mathbb{S}(U_g \bullet U_g^\dagger), \mathbb{S}_{\mathrm{id}}(U_g \bullet U_g^\dagger)\big) = \Big[\langle\langle U_g|^{\otimes 2} \otimes \langle\langle U_g^*|\Big] R_{\mathbb{S}} \Big[|U_g\rangle\rangle^{\otimes 2} \otimes |U_g^*\rangle\rangle\Big] \tag{3.13}$$

for every Quantum Superchannel (actually, for every linear supermap) $\mathbb{S}$, where the two bipartite vectors $|U_g\rangle\rangle$ are in $\mathcal{H}_{\mathrm{out}_1} \otimes \mathcal{H}_{\mathrm{in}_1}$ and in $\mathcal{H}_{\mathrm{out}_2} \otimes \mathcal{H}_{\mathrm{in}_2}$, respectively, and $|U_g^*\rangle\rangle \in \mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}$.

Now, by Eq. (3.10) notice that

$$\begin{aligned} \mu\big(\mathbb{S}_{\mathrm{id}}(U_g \bullet U_g^\dagger), \mathbb{S}_{\mathrm{id}}(U_g \bullet U_g^\dagger)\big) &= \big(R_{U_g^{\otimes 2} \bullet U_g^{\dagger\otimes 2}}, R_{U_g^{\otimes 2} \bullet U_g^{\dagger\otimes 2}}\big) = \\ &= \big(|U_g\rangle\rangle\langle\langle U_g|^{\otimes 2}, |U_g\rangle\rangle\langle\langle U_g|^{\otimes 2}\big) = \\ &= \big[\langle\langle U_g|U_g\rangle\rangle\big]^4 = \\ &= d^4. \end{aligned} \tag{3.14}$$

Then, of course we have

$$\sup_{\mathbb{S}} \mu(\mathbb{S}, \mathbb{S}_{\mathrm{id}}) \leq d^4, \tag{3.15}$$

which suggests to normalize μ introducing

$$F_{\mathbb{S}}(U_g) \doteq \frac{1}{d^4} \mu\big(\mathbb{S}(U_g \bullet U_g^\dagger), \mathbb{S}_{\mathrm{id}}(U_g \bullet U_g^\dagger)\big). \tag{3.16}$$

Notice that $F_{\mathbb{S}}(U_g)$ is the well known Raginsky fidelity [13, 14] between the two output maps $\mathbb{S}(U_g \bullet U_g^\dagger)$ and $\mathbb{S}_{\mathrm{id}}(U_g \bullet U_g^\dagger)$. Evidently[3], $F_{\mathbb{S}}(U_g) \in [0,1]$ and it may be regarded as the fidelity of the supermap $\mathbb{S}$ in the task of cloning the particular unitary transformation U_g: the closer it is to 1, the more the output map $\mathbb{S}(U_g \bullet U_g^\dagger)$ will approximate the desired output $U_g^{\otimes 2} \bullet U_g^{\dagger\otimes 2}$.

Then, of course a Quantum Superchannel $\overline{\mathbb{S}}$ is optimal in the sense of Definition 3.1 if and only if

$$\langle F_{\overline{\mathbb{S}}} \rangle_{\mathbf{G}} = \sup_{\mathbb{S} \in \mathrm{QSC}} \langle F_{\mathbb{S}} \rangle_{\mathbf{G}}, \tag{3.17}$$

where we have introduced the mean fidelity of $\mathbb{S}$ on the group $\mathbf{G}$ as

$$\langle F_{\mathbb{S}} \rangle_{\mathbf{G}} \doteq \int_{\mathbf{G}} \mathrm{d}g \, F_{\mathbb{S}}(U_g). \tag{3.18}$$

3.1.3 Reduction to two-fold Covariant Supermaps

In the present Subsection, we will prove that restricting the search for optimal 1-to-2 unitary cloners to the domain of two-fold Covariant Supermaps does not represent a loss of optimality. So, let us recall the notation introduced in Subsection 2.3.1: we realize that, since all involved Hilbert spaces are isomorphic, we can discard any distinction between unitary representations U, U' (respectively acting on $\mathcal{H}_{\mathrm{in}}$ and $\mathcal{H}_{\mathrm{in}'}$) and V, V' (respectively acting on $\mathcal{H}_{\mathrm{out}}$ and $\mathcal{H}_{\mathrm{out}'}$); furthermore, it is a natural choice to take $U' = U^{\otimes 2}$. With these choices, then, the two equivalent two-fold Covariance conditions (2.76) and (2.79) may be rephrased here respectively as

$$\left[R_{\mathbb{S}}, (U_h \otimes U_g^*)^{\otimes 2} \otimes U_h^* \otimes U_g \right] = 0 \quad \forall\, (g,h) \in \mathbf{G} \times \mathbf{G} \tag{3.19}$$

and

$$\int_{\mathbf{G}} \mathrm{d}g \int_{\mathbf{G}} \mathrm{d}h \left[(U_h \otimes U_g^*)^{\otimes 2} \otimes U_h^* \otimes U_g \right] R_{\mathbb{S}} \left[(U_h \otimes U_g^*)^{\otimes 2} \otimes U_h^* \otimes U_g \right]^\dagger = R_{\mathbb{S}}. \tag{3.20}$$

The first clue leading us to consider two-fold Covariant cloners is the fact that any ideal cloner $\mathbb{S}_{\mathrm{id}}$ is two-fold Covariant, as shown in Figure 3.1 on the facing page.

Furthermore, we have the following

Lemma 3.2 (Invariance of the Mean Fidelity) *The mean cloning fidelity* $\langle F_{\mathbb{S}} \rangle_{\mathbf{G}}$ *is invariant under the two-fold covariance transformation*

$$\mathbb{S} \mapsto \mathbb{V}'_h \mathbb{U}'_g \mathbb{S} \mathbb{V}_{h^{-1}} \mathbb{U}_{g^{-1}} \quad \forall\, (g,h) \in \mathbf{G} \times \mathbf{G}. \tag{3.21}$$

[3] See Footnote 2 on page 62.

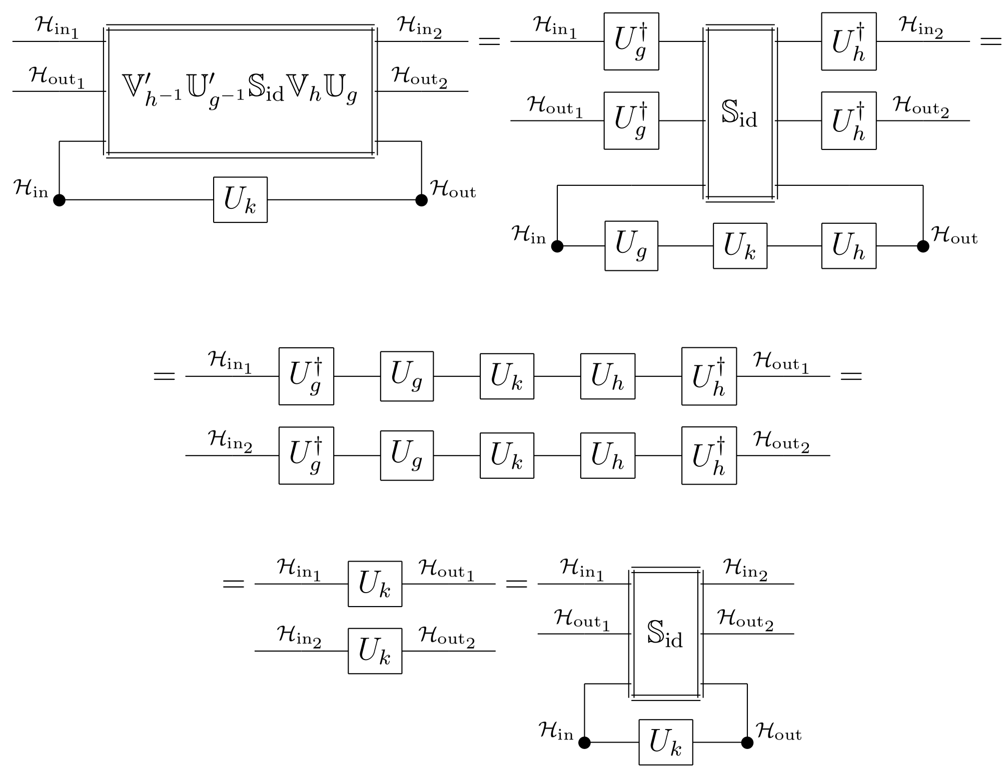


Figure 3.1: This diagramatic equation proves that any ideal cloner $\mathbb{S}_{\text{id}}$ is two-fold Covariant: indeed, the first equality comes from the very definition of supermaps $\mathbb{U}$, $\mathbb{V}$ (see Subsection 2.3.1), the second and the fourth ones come from the definition (3.8) of ideal cloners, and the third is trivial. This proves that $\mathbb{S}_{\text{id}}\mathbb{V}_h\mathbb{U}_g = \mathbb{U}'_g\mathbb{V}'_h\mathbb{S}$ for all $(g, h) \in \mathbf{G} \times \mathbf{G}$, which is exactly the definition of two-fold Covariant Supermaps (see Definition 2.7).

Proof First, we note that

$$F_{\mathbb{S}}(U_g) = F_{[\mathbb{V}'_k \mathbb{U}'_h \mathbb{S} \mathbb{V}_{k^{-1}} \mathbb{U}_{h^{-1}}]}(U_k U_g U_h) \quad \forall\, g, h, k \in \mathbf{G}. \tag{3.22}$$

This comes from from straight calculation, reminding from Eq. (2.70) that

$$R_{[\mathbb{V}'_k \mathbb{U}'_h \mathbb{S} \mathbb{V}_{k^{-1}} \mathbb{U}_{h^{-1}}]} = \left[(U_k \otimes U_h^\top)^{\otimes 2} \otimes U_k^* \otimes U_h^\dagger\right] R_{\mathbb{S}} \left[(U_k \otimes U_h^\top)^{\otimes 2} \otimes U_k^* \otimes U_h^\dagger\right]^\dagger. \tag{3.23}$$

Then, the mean fidelity satisfies

$$\begin{aligned} \langle F_{\mathbb{S}} \rangle_{\mathbf{G}} &= \int_{\mathbf{G}} \mathrm{d}g\, F_{\mathbb{S}}(U_g) = \\ &= \int_{\mathbf{G}} \mathrm{d}g\, F_{[\mathbb{V}'_h \mathbb{U}'_g \mathbb{S} \mathbb{V}_{k^{-1}} \mathbb{U}_{h^{-1}}]}(U_k U_g U_h) = \\ &= \int_{\mathbf{G}} \mathrm{d}g'\, F_{[\mathbb{V}'_k \mathbb{U}'_h \mathbb{S} \mathbb{V}_{k^{-1}} \mathbb{U}_{h^{-1}}]}(U_{g'}) = \langle F_{[\mathbb{V}'_k \mathbb{U}'_h \mathbb{S} \mathbb{V}_{k^{-1}} \mathbb{U}_{h^{-1}}]} \rangle_{\mathbf{G}}, \end{aligned} \tag{3.24}$$

for all $h, k \in \mathbf{G}$ – where we have implicitly put $g' = kgh$. ■

Lemma 3.2 has a very important consequence: let $\overline{\mathbb{S}}$ be *any* optimal cloning Quantum Superchannel. Then, thanks to the Lemma we have that

$$\langle F_{\overline{\mathbb{S}}} \rangle_{\mathbf{G}} = \langle F_{\overline{\mathbb{S}}'(g,h)} \rangle_{\mathbf{G}} \quad \forall\, (g, h) \in \mathbf{G} \times \mathbf{G}, \tag{3.25}$$

for all supermaps $\overline{\mathbb{S}}'(g,h) \doteq \mathbb{V}'_h \mathbb{U}'_g \mathbb{S} \mathbb{V}_{h^{-1}} \mathbb{U}_{g^{-1}}$. Of course, if $\mathbb{S}$ is a Quantum Channel, then $\overline{\mathbb{S}}'(g,h)$ is too: furthermore, it is evident that $\overline{\mathbb{S}}'(g,h)$ is a two-fold Covariant supermap. Then we have proved that, for every optimal cloning Quantum Channel, there exist more two-fold Covariant optimal cloning Quantum Superchannels: this shows that, in the task of optimizing the 1-to-2 unitary cloning, restricting the optimization to the subclass of two-fold Covariant Quantum Superchannels does not represent a loss of optimality.

3.1.4 Explicit Form for the Normalization Condition

In this Subsection we will specialize the general results obtained in Subsections 2.3.3 and 2.3.4, respectively on the isotypic decomposition of covariant supermaps, and on their normalization.

First, consider Eqq. (2.80, 2.81): since in our case we have $U' \otimes U^* = V' \otimes V^* = U^{\otimes 2} \otimes U^*$, then the isotypic decompositions of the input and output spaces are exactly the same, and we rewrite them in a unique form as

$$\mathcal{H}^{\otimes 3} = \bigoplus_{\mu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \bigoplus_{i=1}^{m_\mu} \mathcal{H}_i^{(\mu)}. \tag{3.26}$$

Furthermore, we may rewrite any two-fold Covariant supermap $\mathbb{S}$ as

$$R_{\mathbb{S}} = \sum_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \sum_{i,j=1}^{m_\mu} \sum_{k,l=1}^{m_\nu} r_{\mathbb{S};\mathbb{S};i,j,k,l}^{(\mu,\nu)} T_{j,i}^{(\mu)} \otimes T_{l,k}^{(\nu)*}, \tag{3.27}$$

which is a specialization of Eq. (2.84).

We now consider the TP^2 condition. The following result, strongly depending on Theorem 2.13, allows one to express TP^2 condition as a set of linear conditions on coefficients $r_{\mathbb{S};\mathbb{S};i,j,k,l}^{(\mu,\nu)}$.

Lemma 3.3 (Normalization for Unitary Cloners) *Let $\mathbb{S}$ be a two-fold covariant supermap in the form (3.27), let its input and its output spaces decompose as in Eq. (3.26), and let the input space of its output map decompose as*

$$\mathcal{H}_{\mathrm{in}_1} \otimes \mathcal{H}_{\mathrm{in}_2} = \bigoplus_{\eta=1}^{|\mathrm{Irrep}(U^{\otimes 2})|} \bigoplus_{t=1}^{\tilde{m}_\eta} \tilde{\mathcal{H}}_t^{(\eta)}. \tag{3.28}$$

Furthermore, let $(U,\mathcal{H})$ be irreducible: then, $\mathbb{S}$ is TP^2 if and only if coefficients $\{r_{\mathbb{S};\mathbb{S};i,j,k,l}^{(\mu,\nu)}\}$ satisfy the following $\sum_\eta \tilde{m}_\eta^2$ linear conditions:

$$\sum_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \frac{d_\mu}{d} \sum_{i=1}^{m_\mu} \sum_{k,l=1}^{m_\nu} c_{t,u,l,k}^{(\eta,\nu)} r_{\mathbb{S};\mathbb{S};i,i,k,l}^{(\mu,\nu)} = \delta_{ut} \tilde{d}_\eta \qquad \begin{array}{l} \forall\, \eta = 1,\ldots,|\mathrm{Irrep}(U^{\otimes 2})|, \\ \forall\, u,t = 1,\ldots,\tilde{m}_\eta, \end{array} \tag{3.29}$$

where, for brevity, we have put $d_\mu \doteq \dim(\mathcal{H}^{(\mu)})$, $\tilde{d}_\eta \doteq \dim(\tilde{\mathcal{H}}^{(\eta)})$ and

$$c_{t,u,l,k}^{(\eta,\nu)} \doteq \mathrm{Tr}\left[\tilde{T}_{t,u}^{(\eta)} \mathrm{Tr}_{\mathrm{in}}[T_{l,k}^{(\nu)}]\right]^*. \tag{3.30}$$

Proof Consider Theorem 2.13: since $(U,\mathcal{H})$ is irreducible by hypothesis, we may use TP^2 condition (2.94), that we rewrite here for convenience,

$$\sum_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \frac{d_\mu}{d} \sum_{i=1}^{m_\mu} \sum_{k,l=1}^{m_\nu} r_{\mathbb{S};\mathbb{S};i,i,k,l}^{(\mu,\nu)} \mathrm{Tr}_{\mathrm{in}}[T_{l,k}^{(\nu)*}] = \mathbb{1}_{\mathrm{in}_1} \otimes \mathbb{1}_{\mathrm{in}_2}. \tag{3.31}$$

Now, since $T^{(\nu)}_{l,k}$ is an isometry mapping the irreducible module $\mathcal{H}^{(\nu)}_k$ into the equivalent module $\mathcal{H}^{(\nu)}_l$, then we have the following commutation relation:

$$\begin{aligned} T^{(\nu)}_{l,k} &= [U_g \otimes U_g \otimes U^*_g]|_{\mathcal{H}^{(\nu)}_l} T^{(\nu)}_{l,k} [U_g \otimes U_g \otimes U^*_g]|^{\dagger}_{\mathcal{H}^{(\nu)}_k} \\ &= [U_g \otimes U_g \otimes U^*_g] T^{(\nu)}_{l,k} [U_g \otimes U_g \otimes U^*_g]^{\dagger} \quad \forall\, g \in \mathbf{G}, \end{aligned} \tag{3.32}$$

from which follows

$$\mathrm{Tr}_{\mathrm{in}}[T^{(\nu)}_{k,l}] = [U_g \otimes U_g] \mathrm{Tr}_{\mathrm{in}}[T^{(\nu)}_{l,k}] [U_g \otimes U_g]^{\dagger} \quad \forall\, g \in \mathbf{G}, \tag{3.33}$$

namely $\mathrm{Tr}_{\mathrm{in}}[T^{(\nu)}_{l,k}]$ is an operator on $\mathcal{H}_{\mathrm{in}_1} \otimes \mathcal{H}_{\mathrm{in}_2}$ which commutes with the action $(U^{\otimes 2})$ of $\mathbf{G}$. Then, Theorem A.2 yields

$$\mathrm{Tr}_{\mathrm{in}}[T^{(\nu)}_{l,k}] = \sum_{\eta=1}^{|\mathrm{Irrep}(U^{\otimes 2})|} \sum_{t,u=1}^{\tilde{m}_\eta} \frac{\mathrm{Tr}[\tilde{T}^{(\eta)}_{t,u} \mathrm{Tr}_{\mathrm{in}}[T^{(\nu)}_{l,k}]]}{\tilde{d}_\eta} \tilde{T}^{(\eta)}_{u,t}. \tag{3.34}$$

Substituting in the above Eqq. yields the following TP^2 necessary and sufficient condition:

$$\begin{aligned} \mathbb{1}_{\mathrm{in}_1} \otimes \mathbb{1}_{\mathrm{in}_2} \doteq \sum_{\eta=1}^{|\mathrm{Irrep}(U^{\otimes 2})|} \sum_{t,u=1}^{\tilde{m}_\eta} \sum_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \frac{d_\mu}{d} \sum_{i=1}^{m_\mu} \sum_{k,l=1}^{m_\nu} r^{(\mu,\nu)}_{\mathbb{S};\mathbb{S};i,i,k,l} \cdot \\ \cdot \frac{\mathrm{Tr}[(\tilde{T}^{(\eta)*}_{t,u} \otimes \mathbb{1}_{\mathrm{in}}) T^{(\nu)*}_{l,k}]}{\tilde{d}_\eta} \tilde{T}^{(\eta)*}_{u,t} \end{aligned} \tag{3.35}$$

Now, since $\tilde{T}^{(\eta)}_{t,t}$ are orthogonal projectors on subspaces $\tilde{\mathcal{H}}^{(\eta)}_t$ of $\mathcal{H}_{\mathrm{in}_1} \otimes \mathcal{H}_{\mathrm{in}_2}$, then we have

$$\sum_{\eta=1}^{|\mathrm{Irrep}(U^{\otimes 2})|} \sum_{t=1}^{\tilde{m}_\eta} \tilde{T}^{(\eta)*}_{t,t} = \mathbb{1}_{\mathrm{in}_1} \otimes \mathbb{1}_{\mathrm{in}_2}. \tag{3.36}$$

Furthermore, we realize that an equivalent condition for Eq. (3.35) is obtained by requiring each coefficient of $\tilde{T}^{(\eta)*}_{u,t}$ to be equal to $\delta_{u,t}$: this proves the Theorem. ∎

So far we have been working in the trivial representation where Choi operators of supermaps act on

$$\mathcal{H}^{\otimes 6} \cong \underbrace{\mathcal{H}_{\mathrm{out}_1} \otimes \mathcal{H}_{\mathrm{out}_2}}_{\mathcal{H}_{\mathrm{out}'}} \otimes \underbrace{\mathcal{H}_{\mathrm{in}_1} \otimes \mathcal{H}_{\mathrm{in}_2}}_{\mathcal{H}_{\mathrm{in}'}} \otimes \mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}. \tag{3.37}$$

Now, let us consider the isotypic decomposition of $\mathcal{H}^{\otimes 6}$ under the action of $\mathbf{G} \times \mathbf{G}$:

$$\begin{aligned}
\mathcal{H}^{\otimes 6} &= \bigoplus_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \bigoplus_{i=1}^{m_\mu} \bigoplus_{k=1}^{m_\nu} \mathcal{H}_i^{(\mu)} \otimes \mathcal{H}_k^{(\nu)} \cong \\
&\cong \bigoplus_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \mathcal{H}^{(\mu)} \otimes \mathbb{C}^{m_\mu} \otimes \mathcal{H}^{(\nu)} \otimes \mathbb{C}^{m_\nu} \cong \\
&\cong \bigoplus_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \mathcal{H}^{(\mu)} \otimes \mathcal{H}^{(\nu)} \otimes (\mathbb{C}^{m_\mu} \otimes \mathbb{C}^{m_\nu}),
\end{aligned} \tag{3.38}$$

and let us work in the block-diagonal representation pertaining to the last line. Then, the necessary and sufficient two-fold Covariance condition (3.27) may be rewritten here as

$$R_{\mathbb{S}} = \bigoplus_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \mathbb{1}_{\mathcal{H}^{(\mu)}} \otimes \mathbb{1}_{\mathcal{H}^{(\nu)}} \otimes R_{\mathbb{S}}^{(\mu,\nu)}, \tag{3.39}$$

where each $R_{\mathbb{S}}^{(\mu,\nu)}$ is a square complex matrix of order $m_\mu \cdot m_\nu$ defined in a natural way by

$$\Big[{}_{\mathbb{C}^{m_\mu}}\langle i| \otimes {}_{\mathbb{C}^{m_\nu}}\langle k|\Big] R_{\mathbb{S}}^{(\mu,\nu)} \Big[|j\rangle_{\mathbb{C}^{m_\mu}} \otimes |l\rangle_{\mathbb{C}^{m_\nu}}\Big] = r_{\mathbb{S};i,j,k,l}^{(\mu,\nu)}. \tag{3.40}$$

In a similar way, let us define the square matrix $C^{(\eta,\nu)}$ of order $\tilde{m}_\eta \cdot m_\mu$ by

$$\Big[{}_{\mathbb{C}^{\tilde{m}_\eta}}\langle t| \otimes {}_{\mathbb{C}^{m_\nu}}\langle l|\Big] C^{(\eta,\nu)} \Big[|u\rangle_{\mathbb{C}^{\tilde{m}_\eta}} \otimes |k\rangle_{\mathbb{C}^{m_\nu}}\Big] \doteq c_{t,u,l,k}^{(\eta,\nu)}, \tag{3.41}$$

where $c_{t,u,l,k}^{(\eta,\nu)}$, in turn, are defined in Eq. (3.30). Then we have the following

Corollary 3.4 (to Lemma 3.3) *Under the conditions of Lemma 3.3,* $\mathbb{S}$ *is* TP^2 *if and only if*

$$\mathrm{Tr}[C_{\eta,t,u}^{\oplus} R_{\mathbb{S}}^{\oplus}] = \delta_{ut}\tilde{d}_\eta \qquad \begin{array}{l} \forall\, \eta = 1,\dots,|\mathrm{Irrep}(U^{\otimes 2})|, \\ \forall\, u,t = 1,\dots,\tilde{m}_\eta, \end{array} \tag{3.42}$$

where $C_{\eta,t,u}^{\oplus}$ *and* $R_{\mathbb{S}}^{\oplus}$ *are block-diagonal matrices respectively defined by*

$$C_{\eta,t,u}^{\oplus} \doteq \bigoplus_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \frac{d_\mu}{d} \mathbb{1}_{m_\mu} \otimes \Big(\langle t| \otimes \mathbb{1}_{m_\nu}\Big) C^{(\eta,\nu)} \Big(|u\rangle \otimes \mathbb{1}_{m_\nu}\Big), \tag{3.43}$$

$$R_{\mathbb{S}}^{\oplus} \doteq \bigoplus_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} R_{\mathbb{S}}^{(\mu,\nu)}. \tag{3.44}$$

Proof This comes from straight computation. Indeed, in the block representation, TP^2 condition (3.29) may be rephrased as

$$\begin{aligned}\delta_{ut}\tilde{d}_\eta &\doteq \sum_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \frac{d_\mu}{d}\sum_{i=1}^{m_\mu}\sum_{k,l=1}^{m_\nu}\langle t|\langle l|C^{(\eta,\nu)}|u\rangle|k\rangle\cdot\langle i|\langle k|R_{\mathbb{S}}^{(\mu,\nu)}|i\rangle|l\rangle = \\ &= \sum_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \frac{d_\mu}{d}\sum_{k,l=1}^{m_\nu}\langle l|\Big(\langle t|\otimes \mathbb{1}_{m_\nu}\Big)C^{(\eta,\nu)}\Big(|u\rangle\otimes\mathbb{1}_{m_\nu}\Big)|k\rangle\cdot \\ &\qquad\qquad\cdot\langle k|\mathrm{Tr}_{\mathbb{C}^{m_\mu}}[R_{\mathbb{S}}^{(\mu,\nu)}]|l\rangle = \\ &= \sum_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \frac{d_\mu}{d}\mathrm{Tr}\left[\Big(\langle t|\otimes\mathbb{1}_{m_\nu}\Big)C^{(\eta,\nu)}\Big(|u\rangle\otimes\mathbb{1}_{m_\nu}\Big)\mathrm{Tr}_{\mathbb{C}^{m_\mu}}[R_{\mathbb{S}}^{(\mu,\nu)}]\right] = \\ &= \sum_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \frac{d_\mu}{d}\mathrm{Tr}\left[\left[\mathbb{1}_{m_\mu}\otimes\Big(\langle t|\otimes\mathbb{1}_{m_\nu}\Big)C^{(\eta,\nu)}\Big(|u\rangle\otimes\mathbb{1}_{m_\nu}\Big)\right]R_{\mathbb{S}}^{(\mu,\nu)}\right],\end{aligned} \tag{3.45}$$

for all η, t, u. A further refinement of the last Equation yields the desired result. ■

In the following, we will refer to the block-diagonal complex matrix $R_{\mathbb{S}}^{\oplus}$ as the *reduced Choi operator* of the two-fold Covariant supermap $\mathbb{S}$.

Remark 3.1 Consider Eq. (3.39): then, thanks to Corollary 2.5, it is evident that the corresponding TP^2 supermap $\mathbb{S}$ is $\mathrm{C}^2\mathrm{P}^2$ if and only if $R^{(\mu,\nu)}$ are positive matrices. Equivalently, $\mathbb{S}$ is $\mathrm{C}^2\mathrm{P}^2$ if and only if its reduced Choi operator $R_{\mathbb{S}}^{\oplus}$, defined in Eq. (3.44), is positive. ▲

3.1.5 Reduction to Extremal Supermaps

As we pointed out in Chapter 1, the set[4] QSC of Quantum Superchannels is a convex set: furthermore, in Remark 2.9 we noticed that the set of two-fold Covariant supermaps is an affine space. As a result, the set of two-fold Covariant Quantum Superchannels (which we will be denoting with the symbol $\mathrm{QSC}^{(2\mathrm{f})}$) is, with a very little surprise, a convex set too. This implies that every $\mathbb{S}\in\mathrm{QSC}^{(2\mathrm{f})}$ must admit the convex decomposition

$$\mathbb{S} = \sum_{i\in I} p_i \mathbb{S}_i^{(\mathrm{ext})}, \tag{3.46}$$

[4] In the following, for brevity, we will discard the explicit specifications of Hilbert spaces, e.g. we will write QSC in place of $\mathrm{QSC}(\mathcal{H}_{\mathrm{in}},\mathcal{H}_{\mathrm{out}};\mathcal{H}_{\mathrm{in}'},\mathcal{H}_{\mathrm{out}'})$.

in terms of certain extremal two-fold covariant Quantum Superchannels $\{\mathbb{S}_i^{(\mathrm{ext})} \mid i \in I\}$. Furthermore, since the mean fidelity $\langle F_\bullet \rangle_{\mathbf{G}}$ is linear, then we trivially have

$$\langle F_{\mathbb{S}} \rangle_{\mathbf{G}} = \sum_{i \in I} p_i \langle F_{\mathbb{S}_i^{(\mathrm{ext})}} \rangle_{\mathbf{G}}. \tag{3.47}$$

Now, let us fix an optimal cloning $\overline{\mathbb{S}} \in \mathrm{QSC}^{(2\mathrm{f})}$, i.e. such that[5]

$$\langle F_{\mathbb{S}} \rangle_{\mathbf{G}} \leq \langle F_{\overline{\mathbb{S}}} \rangle_{\mathbf{G}} \quad \forall\, \mathbb{S} \in \mathrm{QSC}^{(2\mathrm{f})}. \tag{3.48}$$

Then, decomposing $\overline{\mathbb{S}}$ into extremal points $\{\overline{\mathbb{S}}_i^{(\mathrm{ext})} \mid i \in \overline{I}\}$ of $\mathrm{QSC}^{(2\mathrm{f})}$, we obtain

$$\langle F_{\overline{\mathbb{S}}} \rangle_{\mathbf{G}} = \sum_{i \in \overline{I}} \overline{p}_i \langle F_{\overline{\mathbb{S}}_i^{(\mathrm{ext})}} \rangle_{\mathbf{G}}, \tag{3.49}$$

so that we must also have

$$\sum_{i \in \overline{I}} \overline{p}_i \langle F_{\overline{\mathbb{S}}_i^{(\mathrm{ext})}} \rangle_{\mathbf{G}} \leq \sum_{i \in \overline{I}} \overline{p}_i \langle F_{\overline{\mathbb{S}}} \rangle_{\mathbf{G}} = \langle F_{\overline{\mathbb{S}}} \rangle_{\mathbf{G}}, \tag{3.50}$$

thanks to the optimality of $\mathbb{S}$, i.e. $\langle F_{\overline{\mathbb{S}}_i^{(\mathrm{ext})}} \rangle_{\mathbf{G}} \leq \langle F_{\overline{\mathbb{S}}} \rangle_{\mathbf{G}}$ for all i: finally, this evidently requires $\langle F_{\overline{\mathbb{S}}_i^{(\mathrm{ext})}} \rangle_{\mathbf{G}} = \langle F_{\overline{\mathbb{S}}} \rangle_{\mathbf{G}}$ for all i.

Thus, if $\overline{\mathbb{S}}$ is an optimal cloning two-fold Covariant Quantum Supermap, we have proved that all the extremal elements $\{\overline{\mathbb{S}}_i^{(\mathrm{ext})} \mid i \in \overline{I}\}$ yielding its convex decomposition are optimal as well: this shows that there is no loss of optimality in restricting the search for optimal two-fold Covariant Quantum Superchannel to the set $\mathrm{Ext}(\mathrm{QSC}^{(2\mathrm{f})})$ of extremal elements in $\mathrm{QSC}^{(2\mathrm{f})}$.

Now, collecting the main results that have been proved in the last Subsections, we realize that there is a natural isomorphism between two-fold covariant Quantum Superchannels and their reduced Choi operators: this is explicitly given by

$$\mathrm{QSC}^{(2\mathrm{f})} \cong \left\{ R_{\mathbb{S}}^{\oplus} \in \Omega\Big(\bigoplus\nolimits_{\mu,\nu} \left[\mathbb{C}^{m_\mu} \otimes \mathbb{C}^{m_\nu}\right]\Big) \,\Big|\, \mathrm{Tr}[C_{\eta,t,u}^{\oplus} R^{\oplus}] = \delta_{u,t} \tilde{d}_\eta \;\forall\, \eta, t, u \right\}, \tag{3.51}$$

where we remind that $\Omega(\mathcal{H})$ denotes the convex cone of positive operators on $\mathcal{H}$, and block-diagonal complex matrices $\{C_{\eta,t,u}^{\oplus}\}$ were defined in Eq. (3.43). Then, in order to study the characterization of extremal two-fold Covariant Quantum Superchannels, one can equivalently study extremal points in the above set of reduced Choi operators — that, for brevity, will be denoted by the symbol $\Omega^{\oplus}$. To this end, let us use the formalism of perturbations.

[5] As in Subsection 3.1.3 we have proved that there is no loss of optimality in restricting the optimization task to two-fold Covariant Quantum Superchannels, we may replace Eq. (3.48) with $\langle F_{\mathbb{S}} \rangle_{\mathbf{G}} \leq \langle F_{\overline{\mathbb{S}}} \rangle_{\mathbf{G}}\ \forall\, \mathbb{S} \in \mathrm{QSC}$ as well.

Definition 3.2 (Perturbation) *Let $R^{\oplus} \in \Omega^{\oplus}$: we will say that the Hermitian operator $Z^{\oplus} \in \bigoplus_{\mu,\nu} \left[\mathbb{C}^{m_\mu} \otimes \mathbb{C}^{m_\nu}\right]$ is a* perturbation *of $R^{\oplus}$ when there exists an $\varepsilon > 0$ such that $R^{\oplus} + tZ^{\oplus} \in \Omega^{\oplus}$ for all $t \in [-\varepsilon, \varepsilon]$.*

It should be clear from the above definition that any element $R^{\oplus} \in \Omega^{\oplus}$ is extremal if and only if it only admits the trivial perturbation $Z^{\oplus} = 0$.

Now we need to rephrase Definition 3.2 in a more convenient form: to this end we give the following Lemma, which is just a restatement of Lemma 11 in [6].

Lemma 3.5 (Support Condition for Perturbations) *Let $R^{\oplus} \in \Omega^{\oplus}$, and let $Z^{\oplus} \in \bigoplus_{\mu,\nu} \left[\mathbb{C}^{m_\mu} \otimes \mathbb{C}^{m_\nu}\right]$ be Hermitian. Then, $Z^{\oplus}$ is a perturbation for $R^{\oplus}$ if and only if*

$$\begin{cases} \mathcal{S}\mathrm{upp}(Z^{\oplus}) \subseteq \mathcal{S}\mathrm{upp}(R^{\oplus}_{\mathbb{S}}), \\ \mathrm{Tr}[C^{\oplus}_{\eta,t,u} Z^{\oplus}] = 0 \quad \forall\, \eta, t, u. \end{cases} \tag{3.52}$$

From this follows another important result (corresponding to Theorem 16 in [6]) that provides a full characterization of extremal two-fold covariant cloners:

Theorem 3.6 (Minimal Support Condition) *Let $R^{\oplus} \in \Omega^{\oplus}$: then $R^{\oplus}$ is extremal if and only if it has the minimal support, namely iff*

$$\mathcal{S}\mathrm{upp}(Q^{\oplus}) \subseteq \mathcal{S}\mathrm{upp}(R^{\oplus}) \Rightarrow Q^{\oplus} = R^{\oplus} \quad \forall\, Q^{\oplus} \in \Omega^{\oplus}. \tag{3.53}$$

This, in turn, allows one to state the following result (see Theorem 17 in [6]):

Theorem 3.7 (Characterization of Extremal Cloners) *Let $\mathbb{S}$ be a two-fold Covariant Quantum Superchannel. Then $\mathbb{S}$ is extremal in the set* $\mathrm{QSC}^{(2\mathrm{f})}$ *if and only if*

$$\bigoplus_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2} \otimes U^*)|} \mathcal{B}\Big(\mathcal{S}\mathrm{upp}(R^{(\mu,\nu)}_{\mathbb{S}})\Big) \subseteq \mathrm{Span}\left\{C_{\eta,t,u}\right\}_{\eta,t,u}, \tag{3.54}$$

where matrices $\{R^{(\mu,\nu)}_{\mathbb{S}}\}$ and $\{C_{\eta,t,u}\}$ were respectively defined in Eqq. (3.40, 3.41).

The last Theorem allows us to place a bound on the ranks of matrices $R^{(\mu,\nu)}$: in fact, taking the dimensions on both sides of Eq. (3.54) yields the following result:

Corollary 3.8 (to Theorem 3.7) *Let $\mathbb{S}$ be a two-fold Covariant Quantum Superchannel. Then, if $\mathbb{S}$ is extremal in* $\mathrm{QSC}^{(2\mathrm{f})}$*, it satisfies*

$$\sum_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} \mathrm{rank}^2\left(R_{\mathbb{S}}^{(\mu,\nu)}\right) \leq \sum_{\eta=1}^{|\mathrm{Irrep}(U^{\otimes 2})|} \tilde{m}_\eta^2. \tag{3.55}$$

In the following, we will use Corollary 3.8 as a necessary condition for a two-fold covariant cloner to be extremal.

3.1.6 Explicit Form of the Mean Fidelity

In the present Subsection, we provide a convenient formula for the mean cloning fidelity of any two-fold Covariant supermap.

Lemma 3.9 (Mean Fidelity Formula) *Let $\mathbb{S}$ be a two-fold Covariant supermap with isotypic decomposition (3.27). Then its mean cloning fidelity may be written as*

$$\langle F_{\mathbb{S}}\rangle_{\mathbf{G}} = \frac{1}{d^4}\sum_{\mu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)|} d_\mu \sum_{i,j=1}^{m_\mu} r_{\mathbb{S};i,j,i,j}^{(\mu,\mu)}, \tag{3.56}$$

where we remind that d_μ is the dimension of the μ-th invariant subspace $\mathcal{H}^{(\mu)}$ of $\mathcal{H}^{\otimes 3}$.

Proof Let us rewrite the mean fidelity under the hypotesis that $\mathbb{S}$ is two-fold Covariant: putting the two-fold covariance condition (3.20) into the expression for the mean fidelity we obtain

$$\begin{aligned}
\langle F_{\mathbb{S}}\rangle_{\mathbf{G}} &= \int_{\mathbf{G}} \mathrm{d}k\, F_{\mathbb{S}}(U_k) = \\
&= \frac{1}{d^4}\int_{\mathbf{G}} \mathrm{d}k\, \mathrm{Tr}\Big[\Big[|U_k\rangle\rangle\langle\langle U_k|^{\otimes 2}\otimes |U_k^*\rangle\rangle\langle\langle U_k^*|\Big] R_{\mathbb{S}}\Big] = \\
&= \frac{1}{d^4}\int_{\mathbf{G}} \mathrm{d}k \int_{\mathbf{G}} \mathrm{d}g \int_{\mathbf{G}} \mathrm{d}h\, \mathrm{Tr}\Big[\Big[|U_k\rangle\rangle\langle\langle U_k|^{\otimes 2}\otimes |U_k^*\rangle\rangle\langle\langle U_k^*|\Big] \\
&\quad \Big[(U_h\otimes U_g^*)^{\otimes 2}\otimes U_h^*\otimes U_g\Big] R_{\mathbb{S}} \Big[(U_h\otimes U_g^*)^{\otimes 2}\otimes U_h^*\otimes U_g\Big]^\dagger\Big] = \\
&= \frac{1}{d^4}\int_{\mathbf{G}} \mathrm{d}k \int_{\mathbf{G}} \mathrm{d}g \int_{\mathbf{G}} \mathrm{d}h\, \mathrm{Tr}\Big[\Big[|U_h^\dagger U_k U_g\rangle\rangle\langle\langle U_h^\dagger U_k U_g|^{\otimes 2}\otimes \\
&\qquad \otimes |U_h^\top U_k^* U_g^*\rangle\rangle\langle\langle U_h^\top U_k^* U_g^*|\Big] R_{\mathbb{S}}\Big].
\end{aligned} \tag{3.57}$$

Now, performing the change of variables $k^{-1}h \mapsto h'$, we obtain

$$\begin{aligned}
\langle F_{\mathbb{S}}\rangle_{\mathbf{G}} &= \frac{1}{d^4}\int_{\mathbf{G}}\mathrm{d}h' \int_{\mathbf{G}}\mathrm{d}g\, \mathrm{Tr}\left[\left[|U_{h'}^{\dagger}U_g\rangle\rangle\langle\langle U_{h'}^{\dagger}U_g|^{\otimes 2}\otimes|U_{h'}^{\top}U_g^{*}\rangle\rangle\langle\langle U_{h'}^{\top}U_g^{*}|\right]R_{\mathbb{S}}\right] = \\
&= \frac{1}{d^4}\int_{\mathbf{G}}\mathrm{d}h' \int_{\mathbf{G}}\mathrm{d}g\, \mathrm{Tr}\Big[|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{\otimes 3} \\
&\qquad \left[(U_{h'}\otimes U_g^{*})^{\otimes 2}\otimes U_{h'}^{*}\otimes U_g\right]R_{\mathbb{S}}\left[(U_{h'}\otimes U_g^{*})^{\otimes 2}\otimes U_{h'}^{*}\otimes U_g\right]^{\dagger} = \\
&= \frac{1}{d^4}\mathrm{Tr}\left[|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{\otimes 3}R_{\mathbb{S}}\right],
\end{aligned} \tag{3.58}$$

where we have used, once again, the two-fold Covariance condition (3.20). Thus we have proved that the mean fidelity for two-fold covariant supermaps is given by

$$\langle F_{\mathbb{S}}\rangle_{\mathbf{G}} = \frac{1}{d^4}\langle\langle\mathbb{1}|^{\otimes 3}R_{\mathbb{S}}|\mathbb{1}\rangle\rangle^{\otimes 3}, \tag{3.59}$$

where the three copies of the maximally entangled vector $|\mathbb{1}\rangle\rangle$ lie in $\mathcal{H}_{\mathrm{out}_1}\otimes\mathcal{H}_{\mathrm{in}_1}$, in $\mathcal{H}_{\mathrm{out}_2}\otimes\mathcal{H}_{\mathrm{in}_2}$, and in $\mathcal{H}_{\mathrm{out}}\otimes\mathcal{H}_{\mathrm{in}}$.

Now, let us rewrite Eq. (3.59) using isotypic decomposition (3.27):

$$\langle F_{\mathbb{S}}\rangle_{\mathbf{G}} = \frac{1}{d^4}\sum_{\mu,\nu=1}^{|\mathrm{Irrep}(U^{\otimes 2}\otimes U^{*})|}\sum_{i,j=1}^{m_\mu}\sum_{k,l=1}^{m_\nu} r^{(\mu,\nu)}_{\mathbb{S};\mathbb{S};i,j,k,l}\langle\langle\mathbb{1}|^{\otimes 3}T^{(\mu)}_{j,i}\otimes T^{(\nu)*}_{l,k}|\mathbb{1}\rangle\rangle^{\otimes 3} \tag{3.60}$$

where $T^{(\mu)}_{j,i}$ is an isometry between subspaces of $\mathcal{H}_{\mathrm{out}_1}\otimes\mathcal{H}_{\mathrm{out}_2}\otimes\mathcal{H}_{\mathrm{out}}$, and $T^{(\nu)}_{l,k}$ is an isometry between subspaces of $\mathcal{H}_{\mathrm{in}_1}\otimes\mathcal{H}_{\mathrm{in}_2}\otimes\mathcal{H}_{\mathrm{in}}$. Now we note that

$$\begin{aligned}
\langle\langle\mathbb{1}|^{\otimes 3}T^{(\mu)}_{j,i}\otimes T^{(\nu)*}_{l,k}|\mathbb{1}\rangle\rangle^{\otimes 3} &= \sum_{i,j,k,l,m,n=1}^{d}\langle i|\langle j|\langle k|T^{(\mu)}_{j,i}|l\rangle|m\rangle|n\rangle\cdot \\
&\qquad\cdot\langle i|\langle j|\langle k|T^{(\nu)*}_{l,k}|l\rangle|m\rangle|n\rangle = \\
&= \mathrm{Tr}\left[T^{(\mu)}_{j,i}T^{(\nu)\dagger}_{l,k}\right] = \\
&= \delta_{\mu,\nu}\cdot\delta_{i,k}\cdot\delta_{j,l}\cdot d_\mu,
\end{aligned} \tag{3.61}$$

where we have used the fact that $T^{(\nu)\dagger}_{l,k} = T^{(\nu)}_{k,l}$. Finally, substituting in the above result yields Eq. (3.56). ■

3.1.7 Summary of the Section

In the following we collect the main results that have been proved in Section 3.1, with the primary aim of providing a brief and self-consistent walkthrough for any specific unitary cloning optimization task:

1. For every optimal Quantum Superchannel there exist several two-fold Covariant Quantum Superchannels achieving the same mean fidelity, so that there is no loss of optimality in restricting the domain of optimization to the set of *two-fold Covariant* Quantum Supermaps (Subsection 3.1.3).

2. Writing Choi operators in the block-diagonal representation, $\mathbb{S}$ is a two-fold covariant supermap if and only if $R_{\mathbb{S}}$ is in the form (3.39).

3. If $\mathbb{S}$ is in such a form, then it is $\mathrm{C^2P^2}$ if and only if all matrices $R_{\mathbb{S}}^{(\mu,\nu)}$ are positive or, equivalently, if and only if so is the reduced Choi operator $R_{\mathbb{S}}^{\oplus}$ defined in Eq. (3.44).

4. $\mathbb{S}$ is $\mathrm{TP^2}$ if and only if its reduced Choi operator $R_{\mathbb{S}}^{\oplus}$ satisfies Eq. (3.42).

5. For every optimal two-fold Covariant Quantum Superchannel in $\mathrm{QSC^{(2f)}}$ there exists at least one extremal point in $\mathrm{Ext(QSC^{(2f)})}$ achieving the same mean fidelity, so that there is no loss of optimality in restricting the domain of optimization to *extremal* two-fold Covariant Quantum Superchannels (Subsection 3.1.5).

6. A necessary condition for $\mathbb{S} \in \mathrm{QSC^{(2f)}}$ to be extremal is given in Eq. (3.55).

7. The mean fidelity of any two-fold covariant supermap $\mathbb{S}$ is given by Eq. (3.56).

With the above results, we are now ready to consider some specific cases of unitary cloning.

3.2 Universal two-fold Covariant Cloning

One of the most natural choices for the set $U_{\mathbf{G}} \doteq \{U_g \mid g \in \mathbf{G}\}$ of input unitary transformations to be cloned is made by requiring $(U, \mathcal{H})$ to be the defining representation of $\mathbf{SU}(d)$. In this way, we will be seeking one Quantum Superchannel that is optimal in the task of cloning *all* unitary transformations[6] on $\mathcal{H}$: as we pointed out before, we expect an ideal cloner (i.e. a Quantum Superchannel – or just a linear supermap – that is able to clone perfectly all unitaries in $U_{\mathbf{G}}$) not to exist, as this would be in contradiction with Theorem 3.1. Then, we will need to apply all the main results that were

[6]Apart from a phase. However, since quantum states are defined *modulo* a phase too, then it is not a restriction to ask all unitary transformations to have $\det = 1$.

proved in Section 3.1 in order to find one extremal two-fold $\mathbf{SU}(d)$-Covariant Quantum Superchannel that is optimal in the sense of Definition 3.1.

3.2.1 Explicit Isotypic Decompositions

All important results in Section 3.1 depend more or less explicitly on isotypic decomposition (3.26) of $\mathcal{H}^{\otimes 3}$ under the action of $U^{\otimes 2} \otimes U^*$, and on isotypic decomposition (3.28) under the action of $\mathcal{H}^{\otimes 2}$ of $U^{\otimes 2}$: then, the first step which is required in order to apply such results is finding explicit forms for these decompositions.

The latter is the very well known Clebsch-Gordan decomposition

$$\mathcal{H}^{\otimes 2} = \mathcal{H}_{\mathrm{sym}} \oplus \mathcal{H}_{\mathrm{ant}}, \tag{3.62}$$

with symmetric and anti-symmetric subspaces

$$\begin{aligned} \mathcal{H}_{\mathrm{sym}} &= \mathrm{Span}\left\{ \left. \frac{|i\rangle \otimes |j\rangle + |j\rangle \otimes |i\rangle}{\sqrt{2}} \right|_{j<i}, |i\rangle \otimes |i\rangle \right\}_i, & (3.63)\\ \mathcal{H}_{\mathrm{ant}} &= \mathrm{Span}\left\{ \left. \frac{|i\rangle \otimes |j\rangle - |j\rangle \otimes |i\rangle}{\sqrt{2}} \right|_{j<i} \right\}_i, & (3.64) \end{aligned}$$

and the obvious projection operators $P_{\mathrm{sym}}, P_{\mathrm{ant}}$.

In order to find the explicit form for isotypic decomposition (3.26), we will use the Young diagrams formalism [15]:

$$\begin{aligned} \overset{d}{\square} \otimes \overset{d}{\square} \otimes \overset{d}{\begin{array}{c}\square\\\square\\\square\\\vdots\\\square\end{array}} &= \left(\overset{\frac{d(d+1)}{2}}{\square\square} \oplus \overset{\frac{d(d-1)}{2}}{\begin{array}{c}\square\\\square\end{array}} \right) \otimes \overset{d}{\begin{array}{c}\square\\\square\\\square\\\vdots\\\square\end{array}} = \\ &= \overset{\frac{d(d+1)}{2}}{\square\square} \otimes \overset{d}{\begin{array}{c}\square\\\square\\\square\\\vdots\\\square\end{array}} \oplus \overset{\frac{d(d-1)}{2}}{\begin{array}{c}\square\\\square\end{array}} \otimes \overset{d}{\begin{array}{c}\square\\\square\\\square\\\vdots\\\square\end{array}}, \end{aligned} \tag{3.65}$$

with

$$
\begin{aligned}
\overset{\frac{d(d+1)}{2}}{\ydiagram{2}} \otimes \overset{d}{\begin{array}{c}\ydiagram{1,1,1}\\ \vdots \\ \ydiagram{1}\end{array}} &= \overset{\frac{(d+2)d(d-1)}{2}}{\begin{array}{l}\ydiagram{3,1,1}\\ \vdots \\ \ydiagram{1}\end{array}} \oplus \overset{d}{\begin{array}{l}\ydiagram{2,1,1}\\ \vdots \\ \ydiagram{1,1}\end{array}} = \\
&= \overset{\frac{(d+2)d(d-1)}{2}}{\begin{array}{l}\ydiagram{3,1,1}\\ \vdots \\ \ydiagram{1}\end{array}} \oplus \overset{d}{\square_1} \ ,
\end{aligned}
\tag{3.66}
$$

and

$$
\begin{aligned}
\overset{\frac{d(d-1)}{2}}{\ydiagram{1,1}} \otimes \overset{d}{\begin{array}{c}\ydiagram{1,1,1}\\ \vdots \\ \ydiagram{1}\end{array}} &= \overset{\frac{(d+1)d(d-2)}{2}}{\begin{array}{l}\ydiagram{2,2,1}\\ \vdots \\ \ydiagram{1}\end{array}} \oplus \overset{d}{\begin{array}{l}\ydiagram{2,1,1}\\ \vdots \\ \ydiagram{1,1}\end{array}} = \\
&= \overset{\frac{(d+1)d(d-2)}{2}}{\begin{array}{l}\ydiagram{2,2,1}\\ \vdots \\ \ydiagram{1}\end{array}} \oplus \overset{d}{\square_2} \ .
\end{aligned}
\tag{3.67}
$$

Summarizing the results of the above diagramatic equations, we have that the isotypic decomposition of $\mathcal{H}^{\otimes 3}$ may be rewritten as

$$
\begin{aligned}
\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H} &= \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \left(\mathcal{H}_1^{(3)} \oplus \mathcal{H}_2^{(3)}\right) \cong \\
&\cong \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus (\mathcal{H}\otimes\mathbb{C}^2)
\end{aligned}
\tag{3.68}
$$

where we have estabilished the following correspondences[7]:

$$
\begin{aligned}
\mathcal{H}^{(1)} &\leftrightarrow \ydiagram{3,1} \ , \\
\mathcal{H}^{(2)} &\leftrightarrow \ydiagram{2,2,1} \ , \\
\mathcal{H}_1^{(3)} &\leftrightarrow \square_1 \ , \\
\mathcal{H}_2^{(3)} &\leftrightarrow \square_2 \ .
\end{aligned}
\tag{3.69}
$$

Then, evidently we also have $|\mathrm{Irrep}(U^{\otimes 2}\otimes U^*)| = 3$, with multiplicites $m_1 = m_2 = 1$ and $m_3 = 2$.

Now, in order to explicitly write down projection operators onto the above spaces, let us rephrase Eq. (3.66) as

$$
\mathcal{H}_{\mathrm{sym}} \otimes \mathcal{H} \cong \mathcal{H}^{(1)} \oplus \mathcal{H}_1^{(3)}. \tag{3.70}
$$

[7]Notice that $\mathcal{H}_1^{(3)} \cong \mathcal{H}_2^{(3)} \cong \mathcal{H}$.

Since it is straightforward to write

$$\mathcal{H}_1^{(3)} = \mathrm{Span}\left\{\frac{\sum_{j=1}^{d}\left[|i\rangle\otimes|j\rangle + |j\rangle\otimes|i\rangle\right]\otimes|j\rangle}{\sqrt{2(d+1)}}\right\}_i, \tag{3.71}$$

then it is straightforward to find projection operators P_{sym}, $T_{1,1}^{(3)}$, and thus $T_{1,1}^{(1)}$, given by

$$T_{1,1}^{(1)} = P_{\mathrm{sym}} \otimes \mathbb{1}_{\mathcal{H}} - T_{1,1}^{(3)}. \tag{3.72}$$

The same procedure may be applied to Eq. (3.67), which reads

$$\mathcal{H}_{\mathrm{ant}} \otimes \mathcal{H} = \mathcal{H}^{(2)} \oplus \mathcal{H}_2^{(3)}. \tag{3.73}$$

Indeed, we have:

$$\mathcal{H}_2^{(3)} = \mathrm{Span}\left\{\frac{\sum_{j=1}^{d}\left[|i\rangle\otimes|j\rangle - |j\rangle\otimes|i\rangle\right]\otimes|j\rangle}{\sqrt{2(d-1)}}\right\}_i, \tag{3.74}$$

yielding explicit forms for P_{ant}, $T_{2,2}^{(3)}$ and $T_{1,1}^{(2)}$, which is given by

$$T_{1,1}^{(2)} = P_{\mathrm{ant}} \otimes \mathbb{1}_{\mathcal{H}} - T_{2,2}^{(3)}. \tag{3.75}$$

It may be easily checked that $T_{1,1}^{(2)}$ is null when $d = 2$.

3.2.2 Covariance, Positivity, Normalization, Extremality

Finally, we are ready to implement the main results that were proved in Section 3.1 (for a brief review, see Subsection 3.1.7). First of all, we remind that $\mathbb{S}$ is covariant iff it is in the form (3.27): then, every two-fold $\mathbf{SU}(d)$-Covariant linear supermap $\mathbb{S}$ depends on at most

$$\sum_{\mu,\nu=1}^{3}\sum_{i,j=1}^{m_\mu}\sum_{k,l=1}^{m_\nu} = 36 \tag{3.76}$$

complex parameters $\{r_{\mathbb{S};i,j,k,l}^{(\mu,\nu)}\}$, regardless the dimension d of the Hilbert spaces.

As we have already seen, C^2P^2 condition amounts to require matrices $R_{\mathbb{S}}^{(\mu,\nu)}$ – defined by Eq. (3.40) – to be positive. Figure 3.2 on the next page explicitly shows such matrices.

$$
\begin{aligned}
R_{\mathbb{S}}^{(1,1)} &= r_{1,1,1,1}^{(1,1)}, & R_{\mathbb{S}}^{(1,2)} &= r_{1,1,1,1}^{(1,2)}, & R_{\mathbb{S}}^{(1,3)} &= \begin{bmatrix} r_{1,1,1,1}^{(1,3)} & r_{1,1,1,2}^{(1,3)} \\ r_{1,1,2,1}^{(1,3} & r_{1,1,2,2}^{(1,3)} \end{bmatrix}, \\
R_{\mathbb{S}}^{(2,1)} &= r_{1,1,1,1}^{(2,1)}, & R_{\mathbb{S}}^{(2,2)} &= r_{1,1,1,1}^{(2,2)}, & R_{\mathbb{S}}^{(2,3)} &= \begin{bmatrix} r_{1,1,1,1}^{(2,3)} & r_{1,1,1,2}^{(2,3)} \\ r_{1,1,2,1}^{(2,3} & r_{1,1,2,2}^{(2,3)} \end{bmatrix}, \\
R_{\mathbb{S}}^{(3,1)} &= \begin{bmatrix} r_{1,1,1,1}^{(3,1)} & r_{1,2,1,1}^{(3,1)} \\ r_{2,1,1,1}^{(3,1)} & r_{2,2,1,1}^{(3,1)} \end{bmatrix}, & R_{\mathbb{S}}^{(3,2)} &= \begin{bmatrix} r_{1,1,1,1}^{(3,2)} & r_{1,2,1,1}^{(3,2)} \\ r_{2,1,1,1}^{(3,2)} & r_{2,2,1,1}^{(3,2)} \end{bmatrix}, & R_{\mathbb{S}}^{(3,3)} &= \begin{bmatrix} r_{1,1,1,1}^{(3,3)} & r_{1,1,1,2}^{(3,3)} & r_{1,2,1,1}^{(3,3)} & r_{1,2,1,2}^{(3,3)} \\ r_{1,1,2,1}^{(3,3)} & r_{1,1,2,2}^{(3,3)} & r_{1,2,2,1}^{(3,3)} & r_{1,2,2,2}^{(3,3)} \\ r_{2,1,1,1}^{(3,3)} & r_{2,1,1,2}^{(3,3)} & r_{2,2,1,1}^{(3,3)} & r_{2,2,1,2}^{(3,3)} \\ r_{2,1,2,1}^{(3,3)} & r_{2,1,2,2}^{(3,3)} & r_{2,2,2,1}^{(3,3)} & r_{2,2,2,2}^{(3,3)} \end{bmatrix}.
\end{aligned}
$$

Figure 3.2: Multiplicity matrices $R_{\mathbb{S}}^{(\mu,\nu)}$ are square matrices of order $m_\mu \cdot m_\nu$.

Furthermore, we have that $(U, \mathcal{H})$ is evidently irreducible, so that we may apply Lemma 3.3: then, $\mathbb{S}$ is TP^2 if and only if it satisfies Eq. (3.29), where coefficients $c^{(\eta,\nu)}_{t,u,l,k}$ are rewritten here as

$$\begin{cases} c^{(1,\nu)}_{1,1,l,k} = \mathrm{Tr}\left[P_{\mathrm{sym}}\mathrm{Tr}_{\mathrm{in}}[T^{(\nu)}_{l,k}]\right]^*, \\ c^{(2,\nu)}_{1,1,l,k} = \mathrm{Tr}\left[P_{\mathrm{ant}}\mathrm{Tr}_{\mathrm{in}}[T^{(\nu)}_{l,k}]\right]^*. \end{cases} \tag{3.77}$$

Now, it is easy to check that

$$\mathrm{Tr}_{\mathrm{in}}[T^{(\nu)}_{l,k}] = \delta_{lk} \cdot d_\nu \cdot \begin{cases} \dfrac{1}{d_{\mathrm{sym}}} P_{\mathrm{sym}} & \text{for } (\nu, k) \in \{(1,1),(3,1)\}, \\ \dfrac{1}{d_{\mathrm{ant}}} P_{\mathrm{ant}} & \text{for } (\nu, k) \in \{(2,1),(3,2)\}, \end{cases} \tag{3.78}$$

so that

$$\mathrm{Tr}\left[P_{\mathrm{sym}}\mathrm{Tr}_{\mathrm{in}}[T^{(\nu)}_{l,k}]\right]^* = \begin{cases} \delta_{lk} \cdot d_\nu & \text{for } (\nu, k) \in \{(1,1),(3,1)\} \\ 0 & \text{for } (\nu, k) \in \{(2,1),(3,2)\} \end{cases}, \tag{3.79}$$

$$\mathrm{Tr}\left[P_{\mathrm{ant}}\mathrm{Tr}_{\mathrm{in}}[T^{(\nu)}_{l,k}]\right]^* = \begin{cases} 0 & \text{for } (\nu, k) \in \{(1,1),(3,1)\} \\ \delta_{lk} \cdot d_\nu & \text{for } (\nu, k) \in \{(2,1),(3,2)\} \end{cases}. \tag{3.80}$$

Thus, TP^2 condition – Eq. (3.29) – may be rewritten as

$$\begin{cases} \displaystyle\sum_{\mu=1}^{3} \frac{d_\mu}{d} \sum_{i=1}^{m_\mu} \sum_{(\nu,k)\in\{(1,1),(3,1)\}} d_\nu r^{(\mu,\nu)}_{\mathbb{S};i,i,k,k} = d_{\mathrm{sym}}, \\ \displaystyle\sum_{\mu=1}^{3} \frac{d_\mu}{d} \sum_{i=1}^{m_\mu} \sum_{(\nu,k)\in\{(2,1),(3,2)\}} d_\nu r^{(\mu,\nu)}_{\mathbb{S};i,i,k,k} = d_{\mathrm{ant}}. \end{cases} \tag{3.81}$$

Finally, we consider the necessary extremality condition, Eq. (3.55). This turns out to be a very strong condition for we have, of course, $|\mathrm{Irrep}(U^{\otimes 2})| = 2$ and $\tilde{m}_1 = \tilde{m}_2 = 1$: then, in this particular case it is equivalent to

$$\sum_{\mu,\nu=1}^{3} \mathrm{rank}^2\left(R^{(\mu,\nu)}_{\mathbb{S}}\right) \leq \sum_{\eta=1}^{2} \tilde{m}^2_\eta = 2. \tag{3.82}$$

Thus, we have proved that a necessary condition for a two-fold $\mathbf{SU}(d)$-covariant Quantum Superchannel $\mathbb{S}$ to be extremal is that $\mathrm{rank}(R^{(\mu,\nu)}_{\mathbb{S}}) \leq 1$ for every μ, ν: using Eq. (3.39), we may thus write

$$R_{\mathbb{S}} = \bigoplus_{\mu,\nu=1}^{3} \mathbb{1}_{\mathcal{H}^{(\mu)}} \otimes \mathbb{1}_{\mathcal{H}^{(\nu)}} \otimes |v^{(\mu,\nu)}_{\mathbb{S}}\rangle\rangle\langle\langle v^{(\mu,\nu)}_{\mathbb{S}}|, \tag{3.83}$$

where $|v_{\mathbb{S}}^{(\mu,\nu)}\rangle\rangle$ are vectors in $\mathbb{C}^{m_\mu} \otimes \mathbb{C}^{m_\nu}$ such that

$$v_{\mathbb{S};i,k}^{(\mu,\nu)} v_{\mathbb{S};j,l}^{(\mu,\nu)*} = r_{\mathbb{S};i,j,k,l}^{(\mu,\nu)}. \tag{3.84}$$

Note thath positivity condition is thus trivially satisfied. Furthermore, notice that extremality condition (3.82) also requires the existance of *at most* two non-zero vectors $|v_{\mathbb{S}}^{(\mu,\nu)}\rangle\rangle$.

3.2.3 Maximization of the Mean Fidelity

Now, let us consider the mean fidelity formula, Eq. (3.56): if $\mathbb{S}$ is an extremal two-fold Covariant Quantum Superchannel, then we may introduce vectors $|v_{\mathbb{S}}^{(\mu,\nu)}\rangle\rangle$ defined in (3.84), so that we obtain

$$\begin{aligned} \langle F_{\mathbb{S}} \rangle_{\mathbf{SU}(d)} &= \frac{1}{d^4} \sum_{\mu=1}^{3} d_\mu \left| \sum_{i=1}^{m_\mu} v_{i,i}^{(\mu,\mu)} \right|^2 = \\ &= \frac{1}{d^4} \left[d_1 |v_{\mathbb{S};1,1}^{(1,1)}|^2 + d_2 |v_{\mathbb{S};1,1}^{(2,2)}|^2 + d_3 |v_{\mathbb{S};1,1}^{(3,3)} + v_{\mathbb{S};2,2}^{(3,3)}|^2 \right]. \end{aligned} \tag{3.85}$$

As we already pointed out, $\mathrm{C}^2\mathrm{P}^2$ condition is trivially satisfied by the choice (3.84), whilst TP^2 condition (3.81) may be rewritten here as

$$\begin{cases} \displaystyle\sum_{\mu=1}^{3} \frac{d_\mu}{d} \sum_{i=1}^{m_\mu} \sum_{(\nu,k) \in \{(1,1),(3,1)\}} d_\nu |v_{\mathbb{S};i,k}^{(\mu,\nu)}|^2 = d_{\mathrm{sym}}, \\ \displaystyle\sum_{\mu=1}^{3} \frac{d_\mu}{d} \sum_{i=1}^{m_\mu} \sum_{(\nu,k) \in \{(2,1),(3,2)\}} d_\nu |v_{\mathbb{S};i,k}^{(\mu,\nu)}|^2 = d_{\mathrm{ant}}. \end{cases} \tag{3.86}$$

We realize that only 4 out of the 16 absolute values $\{|v_{\mathbb{S};i,k}^{(\mu,\nu)}|\}$ do contribute to $\langle F_{\mathbb{S}} \rangle_{\mathbf{SU}(d)}$ and that they all appear in TP^2 condition with a positive sign: then, in order to maximize $\langle F_{\mathbb{S}} \rangle_{\mathbf{SU}(d)}$, we are allowed to set

$$|v_{\mathbb{S};i,k}^{(\mu,\nu)}| = \delta_{\mu,\nu} \delta_{i,k} |v_{\mathbb{S};i,i}^{(\mu,\mu)}|. \tag{3.87}$$

With such a particular choice, TP^2 condition may be rewritten as

$$\begin{cases} d_1^2 |v_{\mathbb{S};1,1}^{(1,1)}|^2 + d_3^2 |v_{\mathbb{S};1,1}^{(3,3)}|^2 = d \cdot d_{\mathrm{sym}}, \\ d_2^2 |v_{\mathbb{S};1,1}^{(2,2)}|^2 + d_3^2 |v_{\mathbb{S};2,2}^{(3,3)}|^2 = d \cdot d_{\mathrm{ant}}. \end{cases} \tag{3.88}$$

Thus, in what follows we shall consider all multiplicity matrices $R_{\mathbb{S}}^{(\mu,\nu)}$ to be zero matrices, except for $R_{\mathbb{S}}^{(1,1)}$, $R_{\mathbb{S}}^{(2,2)}$ and $R_{\mathbb{S}}^{(3,3)}$, that will be supposed to be in the forms

$$R_{\mathbb{S}}^{(1,1)} = |v_{\mathbb{S};1,1}^{(1,1)}|^2, \tag{3.89}$$

$$R_{\mathbb{S}}^{(2,2)} = |v_{\mathbb{S};1,1}^{(2,2)}|^2, \tag{3.90}$$

$$R_{\mathbb{S}}^{(3,3)} = \begin{bmatrix} |v_{\mathbb{S};1,1}^{(3,3)}|^2 & 0 & 0 & v_{\mathbb{S};1,1}^{(3,3)} v_{\mathbb{S};2,2}^{(3,3)*} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ v_{\mathbb{S};1,1}^{(3,3)*} v_{\mathbb{S};2,2}^{(3,3)} & 0 & 0 & |v_{\mathbb{S};2,2}^{(3,3)}|^2 \end{bmatrix}. \tag{3.91}$$

Now, in order to maximize the mean fidelity we are left with six options: either there are two nonzero vectors (in which case they might be $v_{\mathbb{S}}^{(1,1)}$ and $v_{\mathbb{S}}^{(2,2)}$, or $v_{\mathbb{S}}^{(1,1)}$ and $v_{\mathbb{S}}^{(3,3)}$, else $v_{\mathbb{S}}^{(2,2)}$ and $v_{\mathbb{S}}^{(3,3)}$), or there is only one (either $v_{\mathbb{S}}^{(1,1)}$, or $v_{\mathbb{S}}^{(2,2)}$, else $v_{\mathbb{S}}^{(3,3)}$).

We realize immediatly that $v_{\mathbb{S}}^{(1,1)}$ cannot be the only nonzero vector, for in that case TP^2 condition (3.88) cannot be satisfied: the same remark holds for $v_{\mathbb{S}}^{(2,2)}$. On the other hand, let $v_{\mathbb{S}}^{(1,1)}$ be the only zero vector: then, TP^2 condition (3.86) reads

$$\begin{cases} |v_{\mathbb{S};1,1}^{(3,3)}|^2 & = \dfrac{d_{\mathrm{sym}}}{d} \\ & = \dfrac{d+1}{2}, \\ |v_{\mathbb{S};2,2}^{(3,3)}|^2 & = \dfrac{d \cdot d_{\mathrm{ant}} - d_2^2 |v_{\mathbb{S};1,1}^{(2,2)}|^2}{d^2} \\ & = \dfrac{d-1}{2} - \dfrac{1}{4}(d+1)^2(d-2)^2|v_{\mathbb{S};1,1}^{(2,2)}|^2, \end{cases} \tag{3.92}$$

so that it is just a matter of calculation to show that fidelity reads

$$\langle F_{\mathbb{S}} \rangle_{\mathbf{SU}(d)}\Big|_{v^{(1,1)}=0} = \frac{1}{d^3}\Bigg[d - \frac{1}{4}(d^2-3d-2)(d+1)(d-2)|v_{\mathbb{S};1,1}^{(2,2)}|^2 + \\ +2\sqrt{\frac{d+1}{2}\left(\frac{d-1}{2} - \frac{1}{4}(d+1)^2(d-2)^2|v_{\mathbb{S};1,1}^{(2,2)}|^2\right)}\cos(\phi_{\mathbb{S};2,2}^{(3,3)} - \phi_{\mathbb{S};1,1}^{(3,3)})\Bigg], \tag{3.93}$$

where phases $\phi_{\mathbb{S};i,k}^{(\mu,\nu)}$ were introduced for every complex number $v_{\mathbb{S};i,k}^{(\mu,\nu)}$. Then, it is clear that optimality is achived only if $v_{\mathbb{S}}^{(2,2)}$ is a zero vector too, in which case the only nonzero vector is $v_{\mathbb{S}}^{(3,3)}$, and the mean fidelity reads

$$\langle F_{\mathbb{S}} \rangle_{\mathbf{SU}(d)}\Big|_{v_{\mathbb{S}}^{(1,1)}=v_{\mathbb{S}}^{(2,2)}=0} = \frac{1}{d^3}\left[d + \sqrt{d^2-1}\cos(\phi_{\mathbb{S};2,2}^{(3,3)} - \phi_{\mathbb{S};1,1}^{(3,3)})\right]. \tag{3.94}$$

Similarly, setting $v_{\mathbb{S}}^{(2,2)} = 0$ would eventually require us to set $v_{\mathbb{S}}^{(1,1)} = 0$ in order to achieve optimality, so that once again $v_{\mathbb{S}}^{(3,3)}$ would be the only non-zero vector.

Thus, we are left with one more case only, namely the one where $v_{\mathbb{S}}^{(3,3)} = 0$: in this case, $v_{\mathbb{S}}^{(1,1)}$ and $v_{\mathbb{S}}^{(2,2)}$ are fully specified (apart from their phases) by TP^2 condition (3.86), and by some straightforward algebra we obtain

$$\langle F_{\mathbb{S}} \rangle_{\mathbf{SU}(d)} \bigg|_{v^{(3,3)}=0} = 2 \frac{d^2 - 3}{(d+2)(d+1)d^2(d-1)(d-2)}. \tag{3.95}$$

Clearly, this is not the optimal case, as it scales with d^4, contrarily to Eq. (3.94) which scales with d^2.

Then, we have proved that optimality is reached when there is only one nonzero, rank-1 multiplicity matrix $R_{\overline{\mathbb{S}}}^{(3,3)} = |v_{\overline{\mathbb{S}}}^{(3,3)}\rangle\rangle\langle\langle v_{\overline{\mathbb{S}}}^{(3,3)}|$ with

$$v_{\overline{\mathbb{S}}}^{(3,3)} = \begin{bmatrix} \sqrt{\frac{d+1}{2}} e^{\imath\phi} \\ \sqrt{\frac{d-1}{2}} e^{\imath\phi} \end{bmatrix}, \tag{3.96}$$

such that the optimal mean fidelity is given by Eq. (3.94), that we rewrite here as

$$\langle F_{\overline{\mathbb{S}}} \rangle_{\mathbf{SU}(d)} = \frac{d + \sqrt{d^2 - 1}}{d^3}. \tag{3.97}$$

3.2.4 Study on the Optimality

The present Subsection answers the question: How optimal is the optimal cloner? That is to say, we want to compare the optimal mean fidelity (3.97) with the mean fidelity we could achieve with other possible cloning schemes.

Clearly, the worst cloning scheme $\mathbb{S}_{\mathrm{w}}$ one can imagine is the one where the output map $\mathbb{S}(U_g \bullet U_g^\dagger)$ is chosen randomly and independently of the input map. Actually, this is equivalent to the output being fixed, so that we may set

$$\mathbb{S}(U_g \bullet U_g^\dagger) \doteq |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}| \quad \forall\, g \in \mathbf{G}. \tag{3.98}$$

Then, the mean fidelity of such a fixed-output supermap is

$$\begin{aligned} \langle F_{\mathbb{S}_{\mathrm{w}}} \rangle_{\mathbf{G}} &= \frac{1}{d^4} \int_{\mathbf{G}} \mathrm{d}g \, \mathrm{Tr}\left[(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{\otimes 2})(|U_g\rangle\rangle\langle\langle U_g|)^{\otimes 2} \right] = \\ &= \frac{1}{d^4} \int_{\mathbf{G}} \mathrm{d}g \, |\langle\langle\mathbb{1}|U_g\rangle\rangle|^4 = \\ &= \frac{1}{d^4} \int_{\mathbf{G}} \mathrm{d}g \, |\mathrm{Tr}\,[U_g]|^4 = \\ &= \frac{1}{d^4} \int_{\mathbf{G}} \mathrm{d}g \, \left|\mathrm{Tr}\left[U_g^{\otimes 2}\right]\right|^2 . \end{aligned} \tag{3.99}$$

Now, using isotypic decomposition (3.62), we obtain

$$U_g^{\otimes 2} = U_g^{\otimes 2}|_{\mathcal{H}_{\rm sym}} \oplus U_g^{\otimes 2}|_{\mathcal{H}_{\rm ant}}, \tag{3.100}$$

so that

$$\begin{aligned} \left|\mathrm{Tr}\left[U_g^{\otimes 2}\right]\right|^2 &= \left|\mathrm{Tr}\left[U_g^{\otimes 2}|_{\mathcal{H}_{\rm sym}}\right]\right|^2 + \left|\mathrm{Tr}\left[U_g^{\otimes 2}|_{\mathcal{H}_{\rm ant}}\right]\right|^2 + \\ &+ \mathrm{Tr}\left[U_g^{\otimes 2}|_{\mathcal{H}_{\rm sym}}\right] \cdot \mathrm{Tr}\left[U_g^{\otimes 2}|_{\mathcal{H}_{\rm ant}}\right]^* + \\ &+ \mathrm{Tr}\left[U_g^{\otimes 2}|_{\mathcal{H}_{\rm ant}}\right] \cdot \mathrm{Tr}\left[U_g^{\otimes 2}|_{\mathcal{H}_{\rm sym}}\right]^*. \end{aligned} \tag{3.101}$$

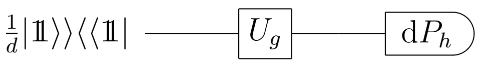
But then, since $\chi_g^{(\rm sym)} = \mathrm{Tr}\left[U_g^{\otimes 2}|_{\mathcal{H}_{\rm sym}}\right]$ and $\chi_g^{(\rm ant)} = \mathrm{Tr}\left[U_g^{\otimes 2}|_{\mathcal{H}_{\rm ant}}\right]$ are the characters of the representation $(U^{\otimes 2}, \mathcal{H}^{\otimes 2})$, then we have

$$\begin{aligned} \left|\mathrm{Tr}\left[U_g^{\otimes 2}\right]\right|^2 &= (\chi_g^{(\rm sym)}, \chi_g^{(\rm sym)}) + (\chi_g^{(\rm ant)}, \chi_g^{(\rm ant)}) + 2\mathbb{R}\mathrm{e}\left[(\chi_g^{(\rm sym)}, \chi_g^{(\rm ant)})\right] = \\ &= 2, \end{aligned} \tag{3.102}$$

thanks to the orthogonality of characters. Thus, substituting in Eq. (3.99) yields

$$\langle F_{\mathbb{S}_{\rm w}} \rangle_{\mathbf{G}} = \frac{2}{d^4}. \tag{3.103}$$

Now, we consider the case in which the unitary to be cloned is estimated, and later re-prepared in two copies (*semi-classical* scheme). In Ref. [16] it is proved that the optimal way to estimate an element U_g of a group $\mathbf{G}$ is achieved by the following physical scheme:



where $\{P_h \mid h \in \mathbf{G}\}$ is a POVM [4] explicitly given by $P_h = d|U_h\rangle\rangle\langle\langle U_h|$ and $\mathrm{d}P_h = P_h \mathrm{d}h$. Now, the probability density of estimating U_h when the input unitary was U_g is given by

$$\mathrm{d}p(h|g) = \mathrm{Tr}\left[\mathrm{d}P_h \frac{|U_g\rangle\rangle\langle\langle U_g|}{d}\right], \tag{3.104}$$

so that our scheme is able to achieve a fidelity which is given by

$$\begin{aligned} F_{\rm s.c.}(U_g) &= \frac{1}{d^4}\int_{\mathbf{SU}(d)} \mathrm{d}p\,(h|g)\mathrm{Tr}\left[(|U_g\rangle\rangle\langle\langle U_g|^{\otimes 2})(|U_h\rangle\rangle\langle\langle U_h|^{\otimes 2})\right] = \\ &= \frac{1}{d^4}\int_{\mathbf{SU}(d)} \mathrm{d}h\ |\langle\langle U_g|U_h\rangle\rangle|^6 \end{aligned} \tag{3.105}$$

and a mean fidelity

$$\begin{aligned}\langle F_{\text{s.c.}}\rangle_{\mathbf{SU}(d)} &= \frac{1}{d^4}\int_{\mathbf{SU}(d)} \mathrm{d}g \int_{\mathbf{SU}(d)} \mathrm{d}h \ |\langle\langle U_g|U_h\rangle\rangle|^6 = \\ &= \frac{1}{d^4}\int_{\mathbf{SU}(d)} \mathrm{d}h \ |\langle\langle \mathbb{1}|U_h\rangle\rangle|^6 = \\ &= \frac{1}{d^4}\int_{\mathbf{SU}(d)} \mathrm{d}h \ |\mathrm{Tr}[U_h]|^6 = \\ &= \frac{1}{d^4}\int_{\mathbf{SU}(d)} \mathrm{d}h \ \left|\mathrm{Tr}[U_h^{\otimes 3}]\right|^2 .\end{aligned} \tag{3.106}$$

Now, using isotypic decomposition (3.68), we obtain

$$U_h^{\otimes 3} = \sum_{\mu=1}^{3}\sum_{i=1}^{m_\mu} U_h^{\otimes 3}|_{\mathcal{H}_i^{(\mu)}}, \tag{3.107}$$

where we remind that $m_1 = 1$, $m_2 = 1$ (apart for the case $d = 2$, where $m_2 = 0$), and $m_3 = 2$. Then,

$$\begin{aligned}\left|\mathrm{Tr}[U_h^{\otimes 3}]\right|^2 &= \left|\sum_{\mu=1}^{3}\sum_{i=1}^{m_\mu} \mathrm{Tr}\left[U_h^{\otimes 3}|_{\mathcal{H}_i^{(\mu)}}\right]\right|^2 = \\ &= \left|\sum_{\mu=1}^{3}\sum_{i=1}^{m_\mu} \chi_{i;g}^{(\mu)}\right|^2 = \\ &= \sum_{\mu=1}^{3}\sum_{i=1}^{m_\mu} \left|\chi_{i;g}^{(\mu)}\right|^2 = \\ &= \begin{cases} 5 \text{ for } d=2, \\ 6 \text{ in all other cases,}\end{cases}\end{aligned} \tag{3.108}$$

once again thanks to the orthogonality of characters. So we conclude with

$$\langle F_{\text{s.c.}}\rangle_{\mathbf{SU}(d)} = \begin{cases} \dfrac{5}{d^4} \text{ for } d=2, \\ \dfrac{6}{d^4} \text{ in all other cases.}\end{cases} \tag{3.109}$$

In Figure 3.3 on the following page the three mean fidelities that have been considered above are plotted for a few values of the dimension d of the Hilbert space $\mathcal{H}$.

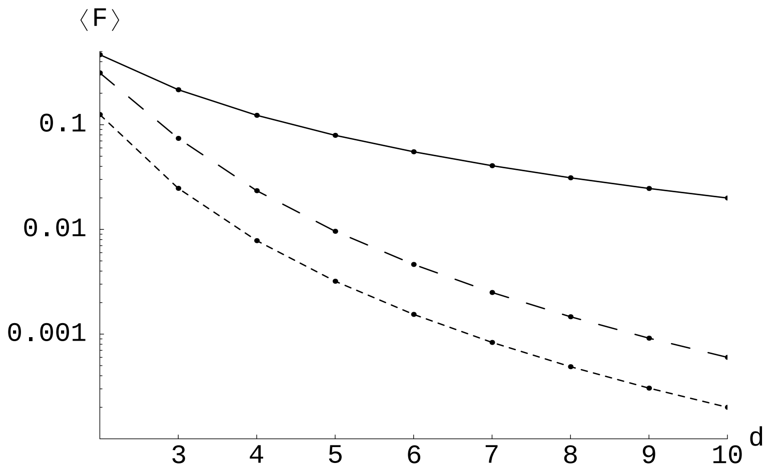


Figure 3.3: The three mean fidelities that have been considered in this Subsection are plotted in a logarithmic scale for a few integer values of the dimension d of the Hilbert space $\mathcal{H}$. Points connected by the solid line correspond to the optimal mean fidelity. Those with long dashes correspond to the semi-classical mean fidelity which is obtained by optimally estimating the unitary and then preparing two copies. Tiny dashes correspond to the fixed-output cloning .

Appendix A

Groups and Representations

This Appendix is not at all to be intended as a review of the theory of groups and representations: for that purpose, there are several excellent books such as [15, 17] to look up. On the contrary, it has the very particular aim of proving Theorem A.2, which is needed for the characterization of covariant supermaps. Then, in Section A.1 we give the definitions and the results that are strictly necessary to prove, in A.2, the Schur Lemma and the subsequent Theorem A.2.

A.1 Definitions and Basic Results

Definition A.1 (Group) *A* group *is a couple* $(\mathbf{G}, \mu)$ *where* $\mathbf{G}$ *is a set of elements* g *and* μ *is an associative composition law* $\mu : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ *admitting one neutral element and the existance of the inverse. It is common use to write* $\mu(g, h) = gh$*, so that in symbols* $(\mathbf{G}, \mu)$ *must satisfy*

$$\begin{cases} g_1(g_2 g_3) = (g_1 g_2) g_3 \quad \forall\, g_1, g_2, g_3 \in \mathbf{G}, \\ \exists\, e \in \mathbf{G} \mid ge = eg = g \quad \forall\, g \in \mathbf{G}, \\ \forall\, g \in \mathbf{G}\ \exists\, g^{-1} \in \mathbf{G} \mid gg^{-1} = g^{-1}g = e. \end{cases} \tag{A.1}$$

In the following, we will also say that $\mathbf{G}$ is a group, thus making the presence of an associative composition law implicit.

A first important distinction between groups is that between *finite* and *continuous* groups, the prototype of the former being the cyclic group of permutation. In the class of continuous groups, a great relevance is given to Lie groups, namely to groups $\mathbf{G}$ that are differentiable manifolds as well: in particular, compact Lie groups such as $\mathbf{U}(d)$ and $\mathbf{SU}(d)$ (respectively, of invertible unitary $d \times d$ matrices, and of unitary $d \times d$ matrices with unitary determinant) share most of their properties with finite groups.

The notion of Represenation of a group is introduced to study the action of the group on a linear space.

Definition A.2 (Representation) *Let* $\mathbf{G}$ *be a group. Then, the couple* (ρ, V) *is a* representation of $\mathbf{G}$ *when* V *is a linear space and* ρ *is an omomorphism of* $\mathbf{G}$ *into the group* $\mathbf{GL}(V) = \mathrm{Aut}(V)$ *of invertible linear operators on* V*, namely when*

$$\rho_g \rho_h = \rho_{gh} \quad \forall\, g, h \in \mathbf{G}. \tag{A.2}$$

In the present treatment, we are interested in the case where V is a finite Hilbert space, and ρ is a unitary representation, namely ρ_g is a unitary operator for all $g \in \mathbf{G}$.

Definition A.3 (Equivalent Representations) *Let* (ρ, V) *and* (ρ', V') *be two representations for* $\mathbf{G}$*. Then, they are said to be* equivalent *when there exists an isomorphism* $T : V \to V'$ *such that*

$$\rho'_g T = T \rho_g \quad \forall\, g \in \mathbf{G}, \tag{A.3}$$

and we will write $V \overset{\mathbf{G}}{\cong} V'$*.*

An omomorphism $T : V \to V'$ is said to be an *intertwining operator* when it satisfies Eq. (A.3), and the linear space of all intertwining operators is commonly denoted with $\mathrm{Hom}_{\mathbf{G}}(V, V')$. Thus, retaining the notation of the previous definition, the two representations are equivalent if and only if $\mathrm{Hom}_{\mathbf{G}}(V, V')$ contains an invertible element. Also notice that, in the case of unitary representations, all invertible intertwiners are isometric operators.

Definition A.4 (Invariant Spaces) *Let* (ρ, V) *be a representation for a group* $\mathbf{G}$*. Then, a subspace* $\mathcal{U}$ *of* V *is said to be* invariant respect to $\mathbf{G}$ *when* $\rho_g(\mathcal{U}) \subseteq \mathcal{U}$ *for all* $g \in \mathbf{G}$*.*

Of course, if we restrict ρ to its action $\rho|_{\mathcal{U}}$ on an invariant subspace $\mathcal{U}$ of V, we have that $(\rho|_{\mathcal{U}}, \mathcal{U})$ is still a representation of $\mathbf{G}$: it is customary to call this a *sub-representation* of (ρ, V).

Definition A.5 (Irreducible Representations) *Let* (ρ, V) *be a representation for a group* $\mathbf{G}$*. Then, it is* irreducible *when the only* $\mathbf{G}$*-invariant subspaces of* V *are the trivial ones.*

Reducible (as opposed to irreducible) representations of finite groups and of compact Lie groups enjoy the very important propriety of *complete reducibility*, namely they may be decomposed into the direct sum of a discrete number of irreducible subrepresentations. In symbols, if we have a reducible representation (ρ, V) of a finite or a compact Lie group, then we may always find a finite set of irreducible subrepresentations $\{(\rho|_{V_k}, V_k)\}$ where

$$V = \bigoplus_k V_k. \tag{A.4}$$

Since it may happen that $V_k \overset{\mathbf{G}}{\cong} V_l$ for some $l \neq k$, then it is customary to rewrite this decomposition as

$$V = \bigoplus_{\mu=1}^{|\mathrm{Irrep}(\rho)|} \sum_{i=1}^{m_\mu} V_i^{(\mu)}, \tag{A.5}$$

where

$$\mathrm{Irrep}(\rho) = \{(\rho|_{V^{(\mu)}}, V^{(\mu)}) \mid V^{(\mu)} \overset{\mathbf{G}}{\ncong} V^{(\nu)}\ \forall\ \mu \neq \nu\} \tag{A.6}$$

denotes the set of the *mutually inequivalent* irreducible representations appearing in Eq. (A.4), and m_μ is the *multiplicity* of the μ-th irreducible representation in decomposition (A.4), so that $V^{(\mu)} \overset{\mathbf{G}}{\cong} V_i^{(\mu)}$ for all $i = 1, \ldots, m_\mu$. Eq. (A.5) is known as *isotypic decomposition* [17].

A.2 Schur Lemma and its Consequences

Theorem A.1 (Schur Lemma) *Let (ρ, V) and (ρ', V') be two irreducible representations of the same group $\mathbf{G}$. Then, if they are equivalent, for all $S \in \mathrm{Hom}_{\mathbf{G}}(V, V')$ we must have $S = \lambda T$, where T is the isomorphic operator introduced in Definition A.3. On the contrary, if they are not equivalent, then* $\mathrm{Hom}_{\mathbf{G}}(V, V') = \{0\}$.

Proof Let $S \in \mathrm{Hom}_{\mathbf{G}}(V, V')$: clearly we have

$$S\rho_g v = \rho'_g S v = 0 \quad \forall\ v \in \mathrm{Ker}(S), \tag{A.7}$$

namely $\rho_g v \in \mathrm{Ker}(S)$ for all $v \in \mathrm{Ker}(S)$ and for all $g \in \mathbf{G}$: then $\mathrm{Ker}(S)$ is an invariant subspace of V. Similarly we have

$$\rho'_g w = \rho'_g S v = S\rho_g v \in \mathrm{Rng}(S) \quad \forall\ w \in \mathrm{Rng}(S), \tag{A.8}$$

namely $\mathrm{Rng}(S)$ is an invariant subspace of V'. On the other hand, the two representations are irreducible by hypothesis, so that the only invariant subspaces must be the trivial ones. In the case $\mathrm{Ker}(S) = V$ and $\mathrm{Rng}(S) = \{0\}$, of course we have $S = 0$, whilst in the case $\mathrm{Ker}(S) = \{0\}$ and $\mathrm{Rng}(S) = V'$ we have that S is an isomorphism, so that the two representations are equivalent: so we have proved that, if the two representations are not equivalent, then $\mathrm{Hom}_{\mathbf{G}}(V, V') = \{0\}$.

Now, let the two representations be equivalent, let T be the isomorphism that was introduced in Definition A.3, and let S be some non-null operator in $\mathrm{Hom}_{\mathbf{G}}(V, V')$; furthermore, let $\lambda \in \mathbb{C}$ be a proper eigenvalue of $T^{-1}S$, namely let $\mathrm{Ker}(T^{-1}S - \lambda I) \neq \{0\}$. Then, it is easy to check that $T^{-1}S$ is an intertwining operator in $\mathrm{Hom}_{\mathbf{G}}(V, V)$, so that the same applies to $T^{-1}S - \lambda \mathbb{1}$, namely

$$[T^{-1}S - \lambda \mathbb{1}]\rho_g = \rho_g[T^{-1}S - \lambda \mathbb{1}] \quad \forall\, g \in \mathbf{G}, \tag{A.9}$$

Now, since we have

$$[T^{-1}S - \lambda \mathbb{1}]\rho_g v = \rho_g[T^{-1}S - \lambda \mathbb{1}]v = 0 \quad \forall\, v \in \mathrm{Ker}(T^{-1}S - \lambda \mathbb{1}), \tag{A.10}$$

then $\rho_g v \in \mathrm{Ker}(T^{-1}S - \lambda \mathbb{1})$ for all $v \in\in \mathrm{Ker}(T^{-1}S - \lambda \mathbb{1})$: this shows that $\mathrm{Ker}(T^{-1}S - \lambda \mathbb{1})$ is an invariant space under the action of $\mathbf{G}$. But since (ρ, V) is irreducible by hypothesis, we have no choice but conclude that $\mathrm{Ker}(T^{-1}S - \lambda \mathbb{1}) = V$. Thus, $T^{-1}Sv = \lambda v$ for all $v \in V$, namely $S = \lambda T$. ■

Finally, we prove the result that allows us to characterize covariant supermaps: as it will be clear by the very proof, it strongly depends on the Schur Lemma.

Theorem A.2 (Characterization of the Commutant) *Let $(U, \mathcal{H})$ be a unitary representation of the group $\mathbf{G}$, and let O be a linear operator on $\mathcal{H}$ commuting with the action of $\mathbf{G}$ via*

$$[U_g, O] = O \quad \forall\, g \in \mathbf{G}. \tag{A.11}$$

Furthermore, let

$$\mathcal{H} \cong \bigoplus_{\mu=1}^{[\mathrm{Irrep}(U)]} \bigoplus_{i=1}^{m_\mu} \mathcal{H}_i^{(\mu)} \tag{A.12}$$

be the isotypic decomposition of $(U, \mathcal{H})$. Then, O admits the following decomposition:

$$O = \sum_{\mu=1}^{[\mathrm{Irrep}(U)]} \sum_{i,j=1}^{m_\mu} \frac{\mathrm{Tr}[T_{i,j}^{(\mu)} O]}{d_\mu} T_{j,i}^{(\mu)}, \tag{A.13}$$

where $T_{j,i}^{(\mu)}$ is any isometry mapping $\mathcal{H}_i^{(\mu)}$ into an equivalent module $\mathcal{H}_j^{(\mu)}$, and d_μ is the dimension of $\mathcal{H}^{(\mu)}$.

Proof Thanks to the definition of isotypic decomposition, each sub-representation $(U|_{\mathcal{H}_i^{(\mu)}}, \mathcal{H}_i^{(\mu)})$ is irreducible. Then, consider a linear application $O_{j,i}^{(\nu,\mu)} : \mathcal{H}_i^{(\mu)} \to \mathcal{H}_j^{(\nu)}$, for some fixed μ, ν, i and j, commuting with the action of $\mathbf{G}$ via

$$U_g|_{\mathcal{H}_j^{(\nu)}} O_{j,i}^{(\nu,\mu)} = O_{j,i}^{(\nu,\mu)} U_g|_{\mathcal{H}_i^{(\mu)}} \quad \forall\, g \in \mathbf{G}. \tag{A.14}$$

Then, thanks to the Schur lemma, it must follow that

$$O_{j,i}^{(\nu,\mu)} = \delta_{\mu\nu} \lambda T_{j,i}^{(\mu)}, \tag{A.15}$$

where λ is a complex number depending on $O_{j,i}^{(\nu,\mu)}$, easily obtained, when $\mu = \nu$, applying $T_{i,j}^{(\mu)} = T_{j,i}^{(\mu)\,-1}$ on the left of the last Eq. and then taking the trace of it, so that

$$\begin{aligned} \mathrm{Tr}\left[T_{i,j}^{(\mu)} O_{j,i}^{(\mu,\mu)}\right] &= \lambda \cdot \mathrm{Tr}\left[T_{i,j}^{(\mu)} T_{j,i}^{(\mu)}\right] \\ &= \lambda \cdot \mathrm{Tr}\left[T_{i,i}^{(\mu)}\right] = \\ &= \lambda \cdot d_\mu, \end{aligned} \tag{A.16}$$

where we have put $d_\mu \doteq \dim(\mathcal{H}^{(\mu)})$. Thus, we have proved that all $\mathbf{G}$-commuting linear applications $O_{j,i}^{(\nu,\mu)}$ of $\mathcal{H}_i^{(\mu)}$ into $\mathcal{H}_j^{(\nu)}$ need to be in the form

$$O_{j,i}^{(\nu,\mu)} = \delta_{\mu\nu} \frac{\mathrm{Tr}\left[T_{i,j}^{(\mu)} O_{j,i}^{(\mu,\mu)}\right]}{d_\mu} T_{j,i}^{(\mu)}. \tag{A.17}$$

Now, consider a $\mathbf{G}$-commuting operator $O \in \mathcal{B}(\mathcal{H})$ as in the statement of the Theorem. Then, since naturally

$$\begin{cases} \displaystyle\sum_{\mu=1}^{[\mathrm{Irrep}(U)]} \sum_{i=1}^{m_\mu} T_{i,i}^{(\mu)} = \mathbb{1}_{\mathcal{H}}, \\ T_{i,i}^{(\mu)} T_{i,i}^{(\mu)} = T_{i,i}^{(\mu)}, \end{cases} \tag{A.18}$$

we may rewrite Eq. (A.11) as

$$\begin{array}{ccc} O & = & U_g O U_g^\dagger \\ \shortparallel & & \shortparallel \\ \displaystyle\sum_{\mu,\nu=1}^{[\mathrm{Irrep}(U)]} \sum_{i=1}^{m_\mu} \sum_{j=1}^{m_\nu} T_{i,i}^{(\mu)} O T_{j,j}^{(\nu)} & & \displaystyle\sum_{\mu,\nu=1}^{[\mathrm{Irrep}(U)]} \sum_{i=1}^{m_\mu} \sum_{j=1}^{m_\nu} U_g T_{i,i}^{(\mu)} T_{i,i}^{(\mu)} O T_{j,j}^{(\nu)} T_{j,j}^{(\nu)} U_g^\dagger \end{array} \tag{A.19}$$

for all g in $\mathbf{G}$. Now, let us define the 'reduced' linear application $O_{i,j}^{(\mu,\nu)}$ of $\mathcal{H}_i^{(\mu)}$ into $\mathcal{H}_j^{(\nu)}$ as

$$O_{j,i}^{(\nu,\mu)} \doteq T_{j,j}^{(\nu)} O T_{i,i}^{(\mu)}. \tag{A.20}$$

Then, since naturally

$$U|_{\mathcal{H}_i^{(\mu)}} \equiv U T_{i,i}^{(\mu)}, \tag{A.21}$$

Eq. (A.19) may be rewritten as

$$\sum_{\mu,\nu=1}^{[\mathrm{Irrep}(U)]} \sum_{i=1}^{m_\mu} \sum_{j=1}^{m_\nu} O_{i,j}^{(\mu,\nu)} = \sum_{\mu,\nu=1}^{[\mathrm{Irrep}(U)]} \sum_{i=1}^{m_\mu} \sum_{j=1}^{m_\nu} U_g|_{\mathcal{H}_i^{(\mu)}} O_{i,j}^{(\mu,\nu)} U_g|_{\mathcal{H}_j^{(\nu)}}^\dagger \quad \forall\, g \in \mathbf{G}. \tag{A.22}$$

Clearly, since terms in the sums are linearly independent, this is equivalent to say that

$$O_{i,j}^{(\mu,\nu)} = U_g|_{\mathcal{H}_i^{(\mu)}} O_{i,j}^{(\mu,\nu)} U_g|_{\mathcal{H}_j^{(\nu)}}^\dagger \quad \forall\, g \in \mathbf{G} \quad \forall\, \mu, \nu, i, j, \tag{A.23}$$

i.e. that all reduced linear applications $O_{i,j}^{(\mu,\nu)}$ must commute with (the action of) $\mathbf{G}$. Then, thanks to the result we proved above, we know they are all in the form (A.17), so that we may write

$$\begin{aligned} O &= \sum_{\mu,\nu=1}^{[\mathrm{Irrep}(U)]} \sum_{i=1}^{m_\mu} \sum_{j=1}^{m_\nu} O_{i,j}^{(\mu,\nu)} = \\ &= \sum_{\mu=1}^{[\mathrm{Irrep}(U)]} \sum_{i,j=1}^{m_\mu} \frac{\mathrm{Tr}\left[T_{i,j}^{(\mu)} O_{j,i}^{(\mu,\mu)}\right]}{d_\mu} T_{j,i}^{(\mu)}, \end{aligned} \tag{A.24}$$

where

$$\begin{aligned} \mathrm{Tr}\left[T_{i,j}^{(\mu)} O_{j,i}^{(\mu,\mu)}\right] &= \mathrm{Tr}\left[T_{i,j}^{(\mu)} T_{j,j}^{(\mu)} O T_{i,i}^{(\mu)}\right] = \\ &= \mathrm{Tr}\left[T_{i,j}^{(\mu)} O\right], \end{aligned} \tag{A.25}$$

as desired. ■

Conclusions

As we already pointed out, the original part of the present work stems from the parallelism between states and Quantum Maps: thus, Quantum Supermaps were axiomatically introduced to describe physical transformations of Quantum Maps, and their axiomatization was carried on in strict analogy with that of Quantum Maps.

Then, a full mathematical characterization of Quantum Supermaps was provided, and it was further proved that there exists a correspondence between Quantum Supermaps and a particular class of physically implementable quantum circuits. Such a result allowed us to consider physical applications of the formalism of supermaps: as an example, the problem of cloning unitary transformations was considered.

Throughout the present work, the hierarchical structure of states, maps and supermaps has been implicitly investigated: indeed, we have shown that states are generalized by maps, and that maps are generalized by supermaps. What distinguishes between them is the normalization condition that they must satisfy; however, even such a condition may be seen as generalizing itself, in passing from states to maps, and then to supermaps.

On the other hand, considering super-supermaps (namely maps of supermaps) does not seem to add any further generality to this hierarchy, as it is straightforward to realize that normalization condition of such super-supermaps would be strictly analogous to that of supermaps, namely super-supermaps may be still regarded as supermaps. This shows that the class of supermaps enjoys a property that we may call *universality*: that is to say, the classes of states and of maps are two subclasses of supermaps, and all generalizations of supermaps are trivial.

It also suggests that a more compact formalism treating equivalently states, maps and supermaps may be favorable, and so motivates a future development of this line of research.

Furthermore, we remark that the supermap formalism is expected to have several applications, of which the cloning of unitaries is just an example. For instance, the issue of programmability (namely, the possibility of performing specific operations that are triggered by programming states) is likely to enjoy important simplifications due to the use of supermaps: suffice it to say that any schematization for programming Quantum Channels using states must be equivalent to some supermap being fed with such programming quantum states – that, as we already pointed out, form a subclass of Quantum Maps.

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addormentata e persa accanto a me, stesa vicino
Quanto sia bello il gioco dell'averti
in sogno verso chissà quale destino.

Francesco Guccini, *Certo non sai*, in *Ritratti*

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