The Quantum Bit Commitment
A complete classification of protocols

Giacomo Mauro D’Ariano
Quantum Optics & Information Group
Istituto Nazionale di Fisica della Materia, Unità di Pavia
Dipartimento di Fisica “A. Volta”, via Bassi 6, I-27100 Pavia, Italy

Dept. of Electrical and Computer Engineering, Northwestern University, Evanston, IL 60208
Dispute: Are there unconditionally secure quantum bit commitment protocols?
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Complete classification of all possible protocols and cheating attacks (ask for preprint or look at quant-ph/ next weeks).
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Bound for the cheating probabilities, for *non-aborting, perfect-verification* protocols.
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Bound for the cheating probabilities, for non-aborting, perfect-verification protocols.
Commitment: provides with a piece of evidence that she has chosen a bit $b = 0, 1$ which she commits to him.
Definition of the problem

Commitment: provides with a piece of evidence that she has chosen a bit \( b = 0, 1 \) which she commits to him.

Opening: Later will open the commitment, revealing \( b \) to , and proving that it is indeed the committed bit with the evidence in Bob’s possession, i.e. will check the committed bit.
Therefore, Alice and Bob should agree on a protocol which satisfies simultaneously the three requirements:

1. The evidence should be concealing, namely should not be able to retrieve before the opening.
2. The evidence should be binding, namely should not be able to change after the commitment.
3. The evidence should be verifiable, namely must be able to check unambiguously against the evidence in his possession.
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Both parties are supposed to possess unlimited technology, and the protocol is said *unconditionally secure* if neither Alice nor Bob can cheat with significant probability of success as a consequence of physical laws.
C. H. Bennet and G. Brassard,
History

History

1984 C. H. Bennet

1993 G. Brassard


1999 R. Jozsa, D. Langlois

2000 H. P. Yuen

The Quantum Bit Commitment: a complete classification of protocols – p.5/28
History

[1984] C. H. Bennet
History

[1984] C. H. Bennet
[2000] H. F. Chau, H. P. Yuen
[2001] C. F. Li and G.-C. Guo

The Quantum Bit Commitment: a complete classification of protocols – p.5/28
Commitment step

prepares the Hilbert space $H$ with the anonymous state $j'$. He then sends $H$ to $\ldots$ modulates the value $b$ of the committed bit on the anonymous state $j'$ and sends the output back to $\ldots$
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Commitment step

1. Prepares the Hilbert space $H$ with the anonymous state $|\varphi\rangle \in H$. He then sends $H$ to $\odot$.

2. Modulates the value $b$ of the committed bit on the anonymous state $|\varphi\rangle$ and sends the output back to $\odot$. 
Bit modulation: QO parametrized by \( b = 0, 1 \).
The most general bit modulation

- Bit modulation: QO parametrized by \( b = 0, 1 \).
- To make the protocol concealing and at the same time verifiable, the modulation is a choice between two ensembles of QO’s \( \{ M_j^{(b)} \} \) for \( b = 0, 1 \) from \( S(H) \) to \( S(K) \).
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- $K \supseteq H$: extending modulation, (e.g. adding decoy systems).
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- \( K \supseteq H \): extending modulation, (e. g. adding decoy systems).

- \( K \subseteq H \): restricting modulation

- \( j \): secret parameter known only to \( K \) parametrizing the choice of different forms for the modulation.
The space of secret parameters has always the option of choosing $j$ by preparing the secret-parameter space $P$ in the state $|j\rangle$ and performing $M^{(b)}$ on $H \otimes P$:

$$M^{(b)} = \sum_j M^{(b)}_j \otimes P_j,$$

where $P_j$ represents the orthonormal projection

$$P_j(\rho) = |j\rangle\langle j| \rho |j\rangle\langle j|.$$
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where \( P_j \) represents the orthonormal projection

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P_j(\rho) = |j\rangle \langle j| \rho |j\rangle \langle j|.
\]

The actually performed QO depends on the state preparation \( \rho_P \) that chooses for the secret-parameter space \( P \):

\[
\text{Tr}_P [M^{(b)} (|\varphi\rangle \langle \varphi| \otimes \rho^{(b)}_P)] = \sum_j M_j^{(b)} (|\varphi\rangle \langle \varphi|) \langle j| \rho^{(b)}_P |j\rangle.
\]
The quantum operations $M_j^{(b)}$ are generally trace-decreasing, i.e. they may be achieved with nonunit probability.
Reduction to trace-preserving

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In terms of the Kraus decomposition

$$M_j^{(b)} (\rho) = \sum_i E_{ji}^{(b)} \rho E_{ji}^{(b)\dag},$$

this means that

$$\sum_i E_{ji}^{(b)\dag} E_{ji}^{(b)} \leq I.$$
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A trace decreasing map is equivalent to a trace preserving one with additional “outcomes” $i$. 
Reduction to unitary

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This can be done as follows

(in the following we will temporarily drop the indices $b$ and $j$).
A trace-preserving QO can be written in the form

\[ M(\rho) = \text{Tr}_F[E \rho E^\dagger], \quad E = \sum_i E_i \otimes |i\rangle \in B(H, K \otimes F) \text{ isometry}. \]
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Unitary embedding of \( H \) into \( K \otimes F \cong H \otimes A \):

\[ E = U(I_H \otimes |\omega\rangle_A), \]
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Unitary embedding of H into \( K \otimes F \cong H \otimes A \):

\[ E = U(I_H \otimes |\omega\rangle_A), \]

we have

\[ M(\rho) = \text{Tr}_F[U(\rho \otimes |\omega\rangle \langle \omega|_A)U^\dagger], \]
Therefore achieves the trace-preserving QO

\[ M(\rho) = \sum_i E_i \rho E_i^\dagger \] knowingly by:

1. preparing an ancilla/decoy state
2. performing a unitary transformation \( U \) on \( H_A \)
3. performing a complete von Neumann measurement on \( F \), with \( K_F \) and outcome \( i \)
4. sending \( K \) to .

Notice that we can have both situations \( H_K \) and \( H_K \), depending on the choice of \( A \) and \( F \).
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\[ M(\rho) = \sum_i E_i \rho E_i^\dagger \text{ knowingly by:} \]

1. preparing an ancilla/decoy state \( |\omega\rangle_A \in A \),
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2. performing a unitary transformation \( U \) on \( H \otimes A \),
3. performing a complete von Neumann measurement on \( F \), with \( K \otimes F \simeq H \otimes A \) and outcome \( i \),
4. sending \( K \) to \( B \).

Notice that we can have both situations \( H \subseteq K \) and \( H \supseteq K \), depending on the choice of \( A \) and \( F \).
Now, if we consider also the preparation of the secret parameter space $P$, the bit commitment step can be achieved as follows:

$$\sum_j p_j^{(b)} M_j^{(b)} (|\varphi\rangle\langle\varphi|) = \sum_j p_j^{(b)} E_{ji}^{(b)} |\varphi\rangle\langle\varphi| E_{ji}^{(b)\dagger}$$

$$= \sum_j p_j^{(b)} \text{Tr}_F[U_j^{(b)} (|\varphi\rangle\langle\varphi| \otimes |\omega\rangle\langle\omega|_A) U_j^{(b)\dagger}] =$$
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$$

where $|\omega\rangle_A$ and $\rho_P$ are independent on $j$ and $b$, and

$$
U^{(b)} = \sum_j U_j^{(b)} \otimes |j\rangle \langle j| \text{ unitary over } H \otimes A \otimes P \simeq K \otimes F \otimes P.
$$
However, for aborting protocols we have:

\[
\sum_j p_j^{(b)} M_j^{(b)} (|\varphi\rangle \langle \varphi|)
\]

\[
= \sum_j p_j^{(b)} \text{Tr}_F [(I_K \otimes \Sigma_{j_F}^{(b)}) U_j^{(b)} (|\varphi\rangle \langle \varphi| \otimes |\omega\rangle \langle \omega|_A) U_j^{(b)\dagger}],
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where \(\Sigma_{j_F}^{(b)}\) denotes an orthonogononal projector on a subspace of \(F\), whose rank generally depends on \(j\) and \(b\).
However, for aborting protocols we have:

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where \(\Sigma_{jF}^{(b)}\) denotes an orthogonal projector on a subspace of \(F\), whose rank generally depends on \(j\) and \(b\).

From now we focus attention on the simplest case of non aborting protocols.
In a perfectly verifiable protocol tells $b$ along with the secret parameter $j$ and the secret outcome $i$ to $A$, who verifies the pure state $E_{ji}^{(b)}|\varphi\rangle$. However, since the local QO's on $K$ and $F$ commute, has the possibility of: (1) first sending $K$ to $A$; (2) then performing the measurement on $F$ at the very last moment of the opening. This is the basis of the EPR cheating attack! However, strictly trace-decreasing QO—i.e. aborting protocols—pose limitations to Alice's EPR cheating, since Alice cannot delay the abortion of the protocol up to the opening, but she must declare it at the commitment.
Opening step

In a perfectly verifiable protocol tells $b$ along with the secret parameter $j$ and the secret outcome $i$ to $M$, who verifies the pure state $E_{ji}^{(b)}|\varphi\rangle$.

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However, strictly trace-decreasing QO—i.e. **aborting protocols**—pose limitations to Alice’s EPR cheating, since Alice cannot delay the abortion of the protocol up to the opening, but she must declare it at the commitment.
Since both secret parameters $j$ and $i$ can be conveniently measured by $\square$, they can be treated on equal footings as a single parameter $J \equiv (j, i)$. 

\[ X_j p_j (b) j M (b) j (j') ih' j E (b) j y; E (b) j (2B (H; K)). \]
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The two maps are then:

$$\sum_j p_j^{(b)} M_j^{(b)} (|\varphi\rangle \langle \varphi|) = \sum_J E_J^{(b)} |\varphi\rangle \langle \varphi| E_J^{(b)\dagger},$$

where $E_J^{(b)} \doteq \sqrt{p_j^{(b)} E_{ji}} \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. 
The principle of *delayed reading*

For non aborting protocols we can reduce a multistep commitment to a single step one, using the principle of delayed reading.
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Principle: Any conditioned QO on H can be regarded as unconditioned on $H \otimes N$ followed by a measurement on N.
The principle of *delayed reading*

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**Principle:** Any conditioned QO on $H$ can be regarded as unconditioned on $H \otimes N$ followed by a measurement on $N$.

1) Bob is requested to make a different QO, say $\{N^{(x)}\}$, depending on the outcome $x$ of previous Alice’s QO.
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2) Bob instead automatizes the conditioned QO, using the un-conditioned one on $H \otimes N$:

$$N = \sum_x N(x) \otimes |x\rangle\langle x|$$
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2) Bob instead *automatizes* the conditioned QO, using the un-conditioned one on $H \otimes N$:

$$N = \sum_x N(x) \otimes |x\rangle\langle x|$$

3) When Bob will measure $N$, the actual QO $N(x)$ will result.
If the knowledge of $x$ is needed only at the opening (non-aborting protocols), then the measurement $|x\rangle\langle x|$ can be delayed up to then.
Reduction to one commitment step

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- Again, each QO can be achieved *knowingly*, by means of a pure measurement.
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For $U_B \in \{U_l\}$, Bob can use instead the unitary $U_B = \sum_l U_l \otimes |l⟩⟨l|$. This is equivalent to another anonymous-state preparation.
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- In conclusion, the whole multi-step protocol is equivalent to a single-step one, with larger spaces $H$, $K$, $A$, $F$, and $P$. 

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Classification of protocols $\equiv$ classifications of QO extensions
Commitment: summary

Classification of protocols ≡ classifications of QO extensions

<table>
<thead>
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### Classification of protocols $\equiv$ classifications of QO extensions

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Cheating!

Pre and post-cheating

\[ \text{can try to cheat by performing a unitary } V \text{ on } F. \text{ This will not change the QO, however, it changes the Kraus decomposition:} \]

\[ f E (b) J g \neq f E (b) J (V) g (\text{same cardinality}) \]

\[ E (b) J (V) = X L E (b) L V L J = h L j V j J i. \]
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post-cheating: can try to cheat by performing a unitary $V$ on $F \otimes P$. This will not change the QO, however, it changes the Kraus decomposition:

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$$\{ E_{j}^{(b)} \} \rightarrow \{ E_{j}^{(b)} (V) \} \text{ (same cardinality)}$$

with

$$E_{j}^{(b)} (V) = \sum_{L} E_{L}^{(b)} V_{LJ}, \quad V_{LJ} = \langle L | V | J \rangle.$$
The probability that can cheat successfully in pretending having committed $b = 1$, whereas she committed $b = 0$ instead, is given by

$$
\overline{P_c^A} = \max_V \int d\mu(\varphi) P_c^A(V, \varphi),
$$
The probability that can cheat successfully in pretending having committed $b = 1$, whereas she committed $b = 0$ instead, is given by

$$\overline{P_c^A} = \max_V \int d\mu(\varphi) P_c^A(V, \varphi),$$

where

$$P_c^A(V, \varphi) = \sum_J \frac{\left| \langle \varphi | E_J^{(0)}(V) \dagger E_J^{(1)} | \varphi \rangle \right|^2}{\left\| E_J^{(1)} \varphi \right\|^2}.$$
Cheating!

A can try to cheat by making the best discrimination between the two maps $M^{(b)} = \sum_j p_j^{(b)} M_j^{(b)}$. 

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Cheating!

- can try to cheat by making the **best discrimination** between the two maps $M^{(b)} = \sum_j p_j^{(b)} M_j^{(b)}$.

- Instead of preparing $|\varphi\rangle \in H$, prepares an entangled state $|\varphi\rangle \in H \otimes R$ and sends only $H$ to .
Cheating!

A can try to cheat by making the **best discrimination** between the two maps \( M^{(b)} = \sum_j p_j^{(b)} M_j^{(b)} \).

Instead of preparing \( |\varphi\rangle \in H \rangle \), prepares an entangled state \( |\varphi\rangle \in H \otimes R \) and sends only \( H \) to .

Cheating probability

\[
P_c^B - \frac{1}{2} \leq \max_{|\varphi\rangle \in H \otimes R} \frac{1}{4} \left\| [M^{(1)} - M^{(0)}] \otimes I_R(|\varphi\rangle\langle\varphi|) \right\|_1 \leq \frac{1}{4} \left\| M^{(1)} - M^{(0)} \right\|_{cb}
\]
Perfectly concealing protocols

\[ \left\| M^{(1)} - M^{(0)} \right\|_{cb} = 0. \]

Then one has \( M^{(1)} = M^{(0)} \)!
Therefore, the two Kraus are connected via a unitary transformation \( V \) on \( F \otimes P \).
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It follows that can cheat with probability one!
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It follows that \( \) can cheat with probability one!

The protocol is not binding!
Approximate concealing

\[ \left\| M^{(1)} - M^{(0)} \right\|_{cb} = \varepsilon, \]

where generally \( \varepsilon \) infinitesimal with \( \dim(K)^{-1} \).
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Problem: is it true that then \( 1 - P_c^A \) is infinitesimal with \( \varepsilon \)?
\[ \| M^{(1)} - M^{(0)} \|_{cb} = \varepsilon, \]

where generally \( \varepsilon \) infinitesimal with \( \dim(K)^{-1} \).

**Problem:** is it true that then \( 1 - \overline{P_c^A} \) is infinitesimal with \( \varepsilon \)?

A affirmative answer would provide the impossibility proof for non aborting protocols.
Bounds for cheating probabilities

\[ P_c^A(V, \varphi) \geq \sqrt{1 - \sum_J \| E_J(0)(V) - E_J(1) \|^2} , \]

\[ \| M^{(1)} - M^{(0)} \|_{cb} \leq \sqrt{\sum_J \| E_J(0)(V) - E_J(1) \|^2} . \]
\[ P_c^A(V, \varphi) \geq \sqrt{1 - \sum_J \left\| E_J^{(0)}(V) - E_J^{(1)} \right\|^2}, \]

\[ \left\| M^{(1)} - M^{(0)} \right\|_{cb} \leq \sqrt{\sum_J \left\| E_J^{(0)}(V) - E_J^{(1)} \right\|^2}. \]

However, is it true that there is a \( V \) such that

\[ \sum_J \left\| E_J^{(0)}(V) - E_J^{(1)} \right\|^2 \leq \omega \left( \left\| M^{(1)} - M^{(0)} \right\|_{cb} \right), \]

with \( \omega(\varepsilon) \) vanishing with \( \varepsilon \)?
For $M^{(1)}$ random unitary, i.e. $E_J^{(1)} = \sqrt{p_J^{(1)}} U_J^{(1)}$ we have

$$[d = \text{dim}(H)]$$

$$P_c^A = \frac{1}{d + 1} + \frac{1}{d(d + 1)} \max V \sum_J \left| \sum_L \text{Tr} \left( U_J^{(1)\dagger} E_L^{(0)} \right) V_{JL} \right|^2.$$
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$$P_c^A = \frac{1}{d+1} + \frac{1}{d(d+1)} \max_V \sum_J \left| \sum_L \text{Tr} \left( U_J^{(1)\dagger} E_L^{(0)} \right) V_{JL} \right|^2.$$ 

An upper bound is given by

$$\frac{1}{d+1} \leq P_c^A \leq \frac{1}{d+1} + \frac{1}{d(d+1)} \|Z\|_1,$$

$$Z_{(JL)K} = \text{Tr}[U_K^{(1)\dagger} E_J^{(0)}] \text{ Tr}[U_K^{(1)} E_L^{(0)\dagger}]$$
Conclusion

There is no general impossibility proof. From the general classification we still don't know if there are proved secure protocols. Bound for cheating probabilities such that:

\[
\begin{align*}
&\text{if violated for all choices of } f(p(b)), \text{ it will provide a secure perfect-verification non-aborting protocol;} \\
&\text{if proved always valid, it would provide an impossibility proof for non-aborting perfect-verification protocols, but we still may have unconditionally secure protocols in the complementary class, e.g. for aborting protocols.}
\end{align*}
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⇒ if violated for all choices of $\{p_{j}^{(b)}\}$, it will provide a secure perfect-verification non-aborting protocol;
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Bound for cheating probabilities such that:

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